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SYNTACTIC COMPLEXITY OF $\mathcal{R}$- AND $\mathcal{J}$-TRIVIAL REGULAR LANGUAGES*

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The syntactic complexity of a subclass of the class of regular languages is the maximal cardinality of syntactic semigroups of languages in that class, taken as a function of the state complexity n of these languages. We prove that $n!$ and $\lfloor e(n-1)! \rfloor$ are tight upper bounds for the syntactic complexity of $\mathcal{R}$- and $\mathcal{J}$-trivial regular languages, respectively.

Keywords: Finite automaton; $\mathcal{J}$-trivial; monoid; regular language; $\mathcal{R}$-trivial; semigroup; syntactic complexity.

1. Introduction

The *state complexity* of a regular language L is the number of states in the minimal deterministic finite automaton (DFA) accepting L. An equivalent notion is *quotient complexity*, which is the number of distinct left quotients of L. The *syntactic complexity* of L is the cardinality of the syntactic semigroup of L. Since the syntactic semigroup of L is isomorphic to the semigroup of transformations performed by the minimal DFA of L, it is natural to consider the relation between syntactic complexity and state complexity. The *syntactic complexity of a subclass of regular languages* is the maximal syntactic complexity of languages in that class, taken as a function of the state complexity of these languages.

Here we consider the classes of languages defined using the well-known Green equivalence relations on semigroups [14]. Let M be a monoid, that is, a semigroup with an identity, and let $s, t \in M$ be any two elements of M. The Green equivalence relations on M, denoted by $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$, are defined as follows: $s \mathcal{L} t \Leftrightarrow Ms = Mt$, $s \mathcal{R} t \Leftrightarrow sM = tM$, $s \mathcal{J} t \Leftrightarrow MsM = MtM$, and $s \mathcal{H} t \Leftrightarrow s \mathcal{L} t$ and $s \mathcal{R} t$. For

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$\rho \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}\}$, M is *ρ-trivial* if and only if $(s,t) \in \rho$ implies $s = t$ for all s, $t \in M$. A language is *ρ-trivial* if and only if its syntactic monoid is ρ-trivial. In this paper we consider only regular ρ-trivial languages. $\mathcal{H}$-trivial regular languages are exactly the star-free languages [14, 17], and $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{J}$-trivial regular languages are all subclasses of the class of star-free languages. The class of $\mathcal{J}$-trivial languages is the intersection of the classes of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial languages.

A language $L \subseteq \Sigma^*$ is *piecewise-testable* if it is a finite boolean combination of languages of the form $\Sigma^* a_1 \Sigma^* \cdots \Sigma^* a_l \Sigma^*$, where $a_i \in \Sigma$. Simon [18, 19] proved in 1972 that a language is piecewise-testable if and only if it is $\mathcal{J}$-trivial. A *biautomaton* is a finite automaton which can read the input word alternatively from left and right. In 2011 Klíma and Polák [10] showed that a language is piecewise-testable if and only if it is accepted by an acyclic biautomaton; here self-loops are allowed, as they are not considered cycles.

In 1979 Brzozowski and Fich [1] proved that a regular language is $\mathcal{R}$-trivial if and only if its minimal DFA is *partially ordered*, that is, it is acyclic as above. They also showed that $\mathcal{R}$-trivial regular languages are finite boolean combinations of languages $\Sigma_1^* a_1 \Sigma_2^* \cdots \Sigma_l^* a_l \Sigma^*$, where $a_i \in \Sigma$ and $\Sigma_i \subseteq \Sigma \setminus \{a_i\}$. Recently Jirásková and Masopust proved a tight upper bound on the state complexity of reversal of $\mathcal{R}$- and $\mathcal{J}$-trivial languages [8, 9].

In the past, the syntactic complexity of the following subclasses of regular languages was considered: In 1970 Maslov [12] noted that n^n was a tight upper bound on the number of transformations performed by a DFA of n states. In 2003–2004, Holzer and König [7], and Krawetz, Lawrence and Shallit [11] studied unary and binary languages. In 2010 Brzozowski and Ye [5] examined ideal and closed regular languages. In 2012 Brzozowski, Li and Ye studied prefix-, suffix-, bifix-, and factor-free regular languages [4]. In 2013 Brzozowski, Li and Liu [3] considered six subclasses of star-free languages including monotonic, partially monotonic, nearly monotonic, finite/cofinite, definite, and reverse definite languages, where L is *definite* (*reverse-definite*) if it can be decided whether a word w belongs to L by examining the suffix (prefix) of w of some fixed length.

We state basic definitions and facts in Sec. 2. In Secs. 3 and 4 we prove tight upper bounds on the syntactic complexities of $\mathcal{R}$- and $\mathcal{J}$-trivial regular languages, respectively. Section 5 concludes the paper. A much shorter version of this work appeared in [2]; many proofs that were omitted there are given in full in the present paper.

2. Preliminaries

Let Q be a non-empty finite set with n elements, and assume without loss of generality that $Q = \{1, 2, \ldots, n\}$. There is a linear order on Q, namely the natural order $<$ on integers. If X is a non-empty subset of Q, then the maximal element in X is denoted by $\max(X)$. A *partition* π of Q is a collection $\pi = \{X_1, X_2, \ldots, X_m\}$ of non-empty subsets of Q such that $Q = X_1 \cup X_2 \cup \cdots \cup X_m$, and $X_i \cap X_j = \emptyset$ for all $1 \leqslant i < j \leqslant m$. We call each subset X_i a *block* of π. For any partition π of Q, let $\text{Max}(\pi) = \{\max(X) \mid X \in \pi\}$. The set of all partitions of Q is denoted by Π_Q. We define a partial order $\preceq$ on Π_Q such that, for any $\pi_1, \pi_2 \in \Pi_Q$, $\pi_1 \preceq \pi_2$ if and only if each block of π_1 is contained in some block of π_2. We say π_1 *refines* π_2 if $\pi_1 \preceq \pi_2$. The poset $(\Pi_Q, \preceq)$ is a finite lattice: For any $\pi_1, \pi_2 \in \Pi_Q$, the *meet* $\pi_1 \wedge \pi_2$ is the $\preceq$-largest partition that refines both π_1 and π_2, and the *join* $\pi_1 \vee \pi_2$ is the $\preceq$-smallest partition that is refined by both π_1 and π_2. From now on, we refer to the lattice $(\Pi_Q, \preceq)$ simply as Π_Q.

A *transformation* of a set Q is a mapping of Q into itself. We consider only transformations t of a finite set Q. If $j \in Q$, then jt is the *image* of j under t. If X is a subset of Q, then $Xt = \{jt \mid j \in X\}$, and the *restriction* of t to X, denoted by $t|_X$, is a mapping from X to Xt such that $jt|_X = jt$ for all $j \in X$. The *composition* of transformations t_1 and t_2 of Q is a transformation $t_1 \circ t_2$ such that $j(t_1 \circ t_2) = (jt_1)t_2$ for all $j \in Q$. We usually drop the operator "$\circ$" and write $t_1 t_2$ for short. An arbitrary transformation can be written in the form

$$t = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ i_1 & i_2 & \cdots & i_{n-1} & i_n \end{pmatrix},$$

where $i_k = kt$, $1 \leqslant k \leqslant n$, and $i_k \in Q$. We also use the notation $t = [i_1, i_2, \ldots, i_n]$ for t above. The *domain* $\text{dom}(t)$ of t is Q. The *range* $\text{rng}(t)$ of t is the set $\text{rng}(t) = Qt$. The *rank* $\text{rank}(t)$ of t is the cardinality of $\text{rng}(t)$, *i.e.*, $\text{rank}(t) = |\text{rng}(t)|$. The binary relation ω_t on $Q \times Q$ is defined as follows: For any $i, j \in Q$, $i \,\omega_t\, j$ if and only if $it^k = jt^l$ for some $k, l \geqslant 0$. This is an equivalence relation, and each equivalence class is called an *orbit* of t. For any $i \in Q$, the orbit of t containing i is denoted by $\omega_t(i)$. The set of all orbits of t is denoted by $\Omega(t)$. Clearly, $\Omega(t)$ is a partition of Q.

A *permutation* of Q is a mapping of Q *onto* itself, so here $\text{rng}(\pi) = Q$. The *identity* transformation $\mathbf{1}$ maps each element to itself. A transformation t is a *cycle* of length k, where $k \geqslant 2$, if there exist pairwise different elements $i_1, \ldots, i_k$ such that $i_1 t = i_2, i_2 t = i_3, \ldots, i_{k-1} t = i_k$, and $i_k t = i_1$, and the remaining elements are mapped to themselves. A cycle is denoted by $(i_1, i_2, \ldots, i_k)$. For $i < j$, a *transposition* is the cycle (i, j). A *unitary* transformation, denoted by $(j \to i)$, has $jt = i$ and $ht = h$ for all $h \neq j$. A *constant* transformation, denoted by $(Q \to i)$, has $jt = i$ for all j. A transformation t is an *idempotent* if $t^2 = t$. The set $\mathcal{T}_Q$ of all transformations of Q is a finite semigroup, in fact, a monoid. We refer the reader to the book of Ganyushkin and Mazorchuk [6] for a detailed discussion of finite transformation semigroups.

For background about regular languages, we refer the reader to [20]. Let Σ be a non-empty finite alphabet. Then Σ^* is the free monoid generated by Σ, and Σ^+ is the free semigroup generated by Σ. A *word* is any element of Σ^*, and the empty word is ε. The length of a word $w \in \Sigma^*$ is $|w|$. A *language* over Σ is any subset of Σ^*. The *reverse of a word* w is denoted by w^R. For a language L, its *reverse* is $L^R = \{w \mid w^R \in L\}$. The *left quotient*, or simply *quotient*, of a language L by a word w is $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$.

The *Myhill congruence* [13] $\approx_L$ of any language L is defined as follows: $x \approx_L y$ if and only if $uxv \in L \Leftrightarrow uyv \in L$ for all $u, v \in \Sigma^*$. This congruence is also known as the *syntactic congruence* of L. The quotient set $\Sigma^+/{\approx_L}$ of equivalence classes of the relation $\approx_L$ is a semigroup called the *syntactic semigroup* of L, and $\Sigma^*/{\approx_L}$ is the *syntactic monoid* of L. The *syntactic complexity* $\sigma(L)$ of L is the cardinality of its syntactic semigroup. A language is regular if and only if its syntactic semigroup is finite. We consider only regular languages, and so assume that all syntactic semigroups and monoids are finite.

A DFA is denoted by $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$, as usual. The DFA $\mathcal{A}$ accepts a word $w \in \Sigma^*$ if $\delta(q_1, w) \in F$. The language accepted by $\mathcal{A}$ is denoted by $L(\mathcal{A})$. If q is a state of $\mathcal{A}$, then the language L_q of q is the language accepted by the DFA $(Q, \Sigma, \delta, q, F)$. Two states p and q of $\mathcal{A}$ are *equivalent* if $L_p = L_q$. If $L \subseteq \Sigma^*$ is a regular language, then its *quotient DFA* is $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$, where $Q = \{w^{-1}L \mid w \in \Sigma^*\}$, $\delta(w^{-1}L, a) = (wa)^{-1}L$, $q_1 = \varepsilon^{-1}L = L$, $F = \{w^{-1}L \mid \varepsilon \in w^{-1}L\}$. The *quotient complexity* $\kappa(L)$ of L is the number of distinct quotients of L. The quotient DFA of L is the minimal DFA accepting L, and so quotient complexity is the same as state complexity.

If $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$ is a DFA, then its *transition semigroup* [14], denoted by $T_{\mathcal{A}}$, consists of all transformations t_w on Q performed by non-empty words $w \in \Sigma^+$ such that $jt_w = \delta(j, w)$ for all $j \in Q$. The syntactic semigroup T_L of a regular language L is isomorphic to the transition semigroup of the quotient DFA $\mathcal{A}$ of L [14], and we represent elements of T_L by transformations in $T_{\mathcal{A}}$. Given a set $G = \{t_a \mid a \in \Sigma\}$ of transformations of Q, we can define the transition function δ of some DFA $\mathcal{A}$ such that $\delta(j, a) = jt_a$ for all $j \in Q$. The transition semigroup of such a DFA is the semigroup generated by G. When the context is clear, we write $a = t$, to mean that the transformation performed by $a \in \Sigma$ is t.

3. $\mathcal{R}$-Trivial Regular Languages

Given DFA $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$, we define the *reachability relation* $\to$ as follows. For all $p, q \in Q$, $p \to q$ if and only if $\delta(p, w) = q$ for some $w \in \Sigma^*$. We say that $\mathcal{A}$ is *partially ordered* [1] if the relation $\to$ is a partial order on Q.

Consider the natural order $<$ on Q. A transformation t of Q is *non-decreasing* if $p \leqslant pt$ for all $p \in Q$. The set $\mathcal{F}_Q$ of all non-decreasing transformations of Q is a semigroup, since the composition of two non-decreasing transformations is again non-decreasing. It was shown in [1] that a language L is $\mathcal{R}$-trivial if and only if its quotient DFA is partially ordered. Equivalently, L is an $\mathcal{R}$-trivial language if and only if its syntactic semigroup contains only non-decreasing transformations.

It is known [6] that $\mathcal{F}_Q$ is generated by the following set

$$\mathcal{GF}_Q = \{\mathbf{1}\} \cup \{t \in \mathcal{F}_Q \mid t^2 = t \text{ and } \operatorname{rank}(t) = n - 1\}.$$

For any transformation t of Q, let $\operatorname{Fix}(t) = \{j \in Q \mid jt = j\}$. Then

Lemma 1. *For any* $t \in \mathcal{GF}_Q$, $\operatorname{rng}(t) = \operatorname{Fix}(t)$.

Proof. Pick arbitrary $t \in \mathcal{GF}_Q$. The claim holds trivially for $\mathbf{1}$. Assume $t \neq \mathbf{1}$. Clearly $\text{Fix}(t) \subseteq \text{rng}(t)$. Suppose there exists $j \in \text{rng}(t)$ but $jt \neq j$. Then $ht = j$ for some $h \in Q$, and $h \neq j$. However, since $ht^2 = jt \neq j = ht$, t is not an idempotent, which is a contradiction. Therefore $\text{rng}(t) = \text{Fix}(t)$. □

If $n = 1$, then $\mathcal{F}_Q$ contains only the identity transformation $\mathbf{1}$, and $\mathcal{GF}_Q = \mathcal{F}_Q = \{\mathbf{1}\}$. So $|\mathcal{GF}_Q| = |\mathcal{F}_Q| = 1$. Let $\binom{n}{m}$ be the binomial coefficient. If $n \geqslant 2$, then we have

Lemma 2. *For $n \geqslant 2$, $|\mathcal{GF}_Q| = 1 + \binom{n}{2}$.*

Proof. Pick $t \in \mathcal{GF}_Q$ such that $t \neq \mathbf{1}$. Then $\text{rank}(t) = n - 1$, and, by Lemma 1, $|\text{Fix}(t)| = n - 1$. There is only one element $j \in Q \setminus \text{Fix}(t)$, and $j < jt$. Note that t is fully determined by the pair (j, jt). Hence there are $\binom{n}{2}$ different t. Together with the identity $\mathbf{1}$, the cardinality of $\mathcal{GF}_Q$ is $1 + \binom{n}{2}$. □

Lemma 3. *If $G \subseteq \mathcal{F}_Q$ and G generates $\mathcal{F}_Q$, then $\mathcal{GF}_Q \subseteq G$.*

Proof. Suppose there exists $t \in \mathcal{GF}_Q$ such that $t \notin G$. Since G generates $\mathcal{F}_Q$, t can be written as $t = g_1 \cdots g_k$ for some $g_1, \dots, g_k \in G$, where $k \geqslant 2$. Then $\text{rng}(g_k) \supseteq \text{rng}(g_{k-1}g_k) \supseteq \cdots \supseteq \text{rng}(g_1 \cdots g_{k-1}g_k) = \text{rng}(t)$. Note that $\mathbf{1}$ is the only element in $\mathcal{F}_Q$ with range Q; so if $t = \mathbf{1}$, then $g_1 = \cdots = g_k = \mathbf{1}$, a contradiction.

Assume $t \neq \mathbf{1}$, and $g_i \neq \mathbf{1}$ for all $i = 1, \dots, k$. Then $\text{rank}(t) = n - 1$, and $\text{rng}(g_1) = \cdots = \text{rng}(g_k) = \text{rng}(t)$. Since each g_i is non-decreasing, for all $p \in \text{Fix}(t)$, we must have $p \in \text{Fix}(g_i)$ as well; so $\text{Fix}(t) \subseteq \text{Fix}(g_i)$. Moreover, since $\text{Fix}(g_i) \subseteq \text{rng}(g_i) = \text{rng}(t)$ and $\text{rng}(t) = \text{Fix}(t)$ by Lemma 1, $\text{Fix}(g_i) = \text{Fix}(t) = \text{rng}(t)$. Now, let q be the unique element in $Q \setminus \text{Fix}(t)$. Then $qg_1 \neq q$, and $qg_1 \in \text{Fix}(g_2) = \cdots = \text{Fix}(g_k)$. So $q(g_1 \cdots g_k) = qg_1$. However, since $t = g_1 \cdots g_k$, $q(g_1 \cdots g_k) = qt$ and $qt = qg_1$. Hence $t = g_1$, and we get a contradiction again. Therefore $\mathcal{GF}_Q \subseteq G$. □

Consequently, $\mathcal{GF}_Q$ is the unique minimal generator of $\mathcal{F}_Q$. We also have

Lemma 4. *For $n \geqslant 1$, $|\mathcal{F}_Q| = n!$.*

Proof. Pick an arbitrary $t \in \mathcal{F}_Q$. For each $p \in Q$, since $p \leqslant pt$, pt can be chosen from $\{p, p+1, \dots, n\}$. Hence $|\mathcal{F}_Q| = n!$. □

Using the lemmas, we obtain our first tight upper bound:

Theorem 5. *If $L \subseteq \Sigma^*$ is an $\mathcal{R}$-trivial regular language of quotient complexity $\kappa(L) = n \geqslant 1$, then its syntactic complexity $\sigma(L)$ satisfies $\sigma(L) \leqslant n!$, and this bound is tight if $|\Sigma| \geqslant 1$ for $n = 1$ and if $|\Sigma| \geqslant 1 + \binom{n}{2}$ for $n \geqslant 2$.*

Proof. Let $\mathcal{A}$ be the quotient DFA of L, and let T_L be its syntactic semigroup. Then T_L is a subset of $\mathcal{F}_Q$, and $\sigma(L) \leqslant n!$.

When $n = 1$, the only regular languages are Σ^* or $\emptyset$, and they are both $\mathcal{R}$-trivial and meet the bound 1. To see the bound is tight for $n \geqslant 2$, let $\mathcal{A}_n = (Q, \Sigma, \delta, 1, \{n\})$ be the DFA with alphabet Σ of size $1 + \binom{n}{2}$ and set of states $Q = \{1, \ldots, n\}$, where each $a \in \Sigma$ defines a distinct transformation in $\mathcal{GF}_Q$. For each $p \in Q$, let $t_p = [p, n, \ldots, n]$. Since $\mathcal{GF}_Q$ generates $\mathcal{F}_Q$ and $t_p \in \mathcal{F}_Q$, $t_p = e_1 \cdots e_k$ for some $e_1, \ldots, e_k \in \mathcal{GF}_Q$, where k depends on p. Then there exist $a_1, \ldots, a_k \in \Sigma$ such that each a_i performs e_i and state p is reached by $w = a_1 \cdots a_k$. Moreover, n is the only final state of $\mathcal{A}_n$. Consider any non-final state $q \in Q \setminus \{n\}$. Since $t = [2, 3, \ldots, n, n] \in \mathcal{F}_Q$, there exist $b_1, \ldots, b_l \in \Sigma$ such that the word $u = b_1 \cdots b_l$ performs t. State q can be distinguished from other non-final states by the word u^{n-q}. Hence $L = L(\mathcal{A}_n)$ has quotient complexity $\kappa(L) = n$. The syntactic monoid of L is $\mathcal{F}_Q$, and so $\sigma(L) = n!$. By Lemma 3, the alphabet of $\mathcal{A}_n$ is minimal. □

Example 6. *When $n = 4$, there are $4! = 24$ non-decreasing transformations of $Q = \{1, 2, 3, 4\}$. Among them, there are 11 transformations with rank $n - 1 = 3$. The following 6 transformations from the 11 are idempotents: $e_1 = [1, 2, 4, 4]$, $e_2 = [1, 3, 3, 4]$, $e_3 = [1, 4, 3, 4]$, $e_4 = [2, 2, 3, 4]$, $e_5 = [3, 2, 3, 4]$, $e_6 = [4, 2, 3, 4]$.*

Together with the identity transformation $\mathbf{1}$, we have the generating set $\mathcal{GF}_Q$ for $\mathcal{F}_Q$ with 7 transformations. We can then define the DFA $\mathcal{A}_4$ with 7 inputs as in the proof of Theorem 5; $\mathcal{A}_4$ is shown in Fig. 1. The quotient complexity of $L = L(\mathcal{A}_4)$ is 4, and the syntactic complexity of L is 24. ■

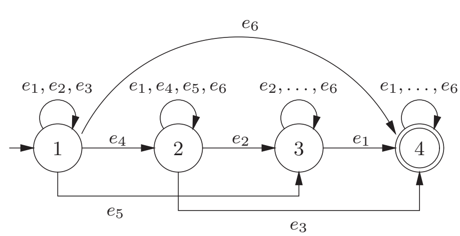

Fig. 1. DFA $\mathcal{A}_4$ with $\kappa(L(\mathcal{A}_4)) = 4$ and $\sigma(L(\mathcal{A}_4)) = 24$; the input performing the identity transformation is not shown.

4. $\mathcal{J}$-Trivial Regular Languages

For any $m \geqslant 1$, we define an equivalence relation $\leftrightarrow_m$ on Σ^* as follows. For any $u, v \in \Sigma^*$, $u \leftrightarrow_m v$ if and only if for every $x \in \Sigma^*$ with $|x| \leqslant m$, x is a subword of u if and only if x is a subword of v. Let L be any language over Σ. Then L is *piecewise-testable* if there exists $m \geqslant 1$ such that, for every $u, v \in \Sigma^*$, $u \leftrightarrow_m v$ implies that $u \in L \Leftrightarrow v \in L$. Let $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$ be a DFA. If Γ is a subset of Σ, a *component* of $\mathcal{A}$ restricted to Γ is a minimal subset P of Q such that, for all $p \in Q$ and $w \in \Gamma^*$, $\delta(p, w) \in P$ if and only if $p \in P$. A state q of $\mathcal{A}$ is *maximal* if $\delta(q, a) = q$ for all $a \in \Sigma$. Simon [19] proved the following characterization of piecewise-testable languages.

Theorem 7 (Simon). *Let L be a regular language over Σ, let $\mathcal{A}$ be its quotient DFA, and let T_L be its syntactic monoid. Then the following are equivalent:*

(1) *L is piecewise-testable.*
(2) *$\mathcal{A}$ is partially ordered, and for every non-empty subset Γ of Σ, each component of $\mathcal{A}$ restricted to Γ has exactly one maximal state.*
(3) *T_L is $\mathcal{J}$-trivial.*

Consequently, a regular language is piecewise-testable if and only if it is $\mathcal{J}$-trivial. The following characterization of $\mathcal{J}$-trivial monoids is due to Saito [16].

Theorem 8 (Saito). *Let S be a monoid of transformations of Q. Then the following are equivalent:*

(1) *S is $\mathcal{J}$-trivial.*
(2) *S is a subset of $\mathcal{F}_Q$ and $\Omega(ts) = \Omega(t) \vee \Omega(s)$ for all $t, s \in S$.*

Let L be a $\mathcal{J}$-trivial language with quotient DFA $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$ and syntactic monoid T_L. Since $T_L \subseteq \mathcal{F}_Q$, an upper bound on the cardinality of $\mathcal{J}$-trivial submonoids of $\mathcal{F}_Q$ is an upper bound on the syntactic complexity of L.

Lemma 9. *If $t, s \in \mathcal{F}_Q$, then*

(1) $\mathrm{Fix}(t) = \mathrm{Max}(\Omega(t))$.
(2) $\Omega(t) \preceq \Omega(s)$ *implies* $\mathrm{Fix}(t) \supseteq \mathrm{Fix}(s)$, *where* $\mathrm{Fix}(t) = \mathrm{Fix}(s)$ *if and only if* $\Omega(t) = \Omega(s)$.

Proof. (1) First, for each $j \in \mathrm{Max}(\Omega(t))$, since $t \in \mathcal{F}_Q$, we have $jt = j$, and $j \in \mathrm{Fix}(t)$. So $\mathrm{Max}(\Omega(t)) \subseteq \mathrm{Fix}(t)$. On the other hand, if there exists $j \in \mathrm{Fix}(t) \setminus \mathrm{Max}(\Omega(t))$, then $jt = j$, and $j < \max(\omega_t(j))$. Let $i = \max(\omega_t(j))$; then $it = i$ and, for any $k, l \geqslant 0$, $jt^k = j < i = it^l$. So $i \notin \omega_t(j)$, which is a contradiction. Hence $\mathrm{Fix}(t) = \mathrm{Max}(\Omega(t))$.

(2) Assume $\Omega(t) \preceq \Omega(s)$. By definition, we have $\mathrm{Max}(\Omega(t)) \supseteq \mathrm{Max}(\Omega(s))$. Then, by 1, $\mathrm{Fix}(t) \supseteq \mathrm{Fix}(s)$. Furthermore, $\Omega(t) = \Omega(s)$ if and only if $\mathrm{Max}(\Omega(t)) = \mathrm{Max}(\Omega(s))$, and if and only if $\mathrm{Fix}(t) = \mathrm{Fix}(s)$. □

Example 10. *Consider non-decreasing transformation $t = [1, 3, 3, 5, 6, 6]$, as shown in Fig. 2(a). The orbit set $\Omega(t)$ has three blocks: $\{1\}$, $\{2, 3\}$, and $\{4, 5, 6\}$. Note that $\mathrm{Fix}(t) = \{1, 3, 6\} = \mathrm{Max}(\Omega(t))$, as expected.*

Let $s = [4, 3, 3, 6, 6, 6]$ be another non-decreasing transformation, as shown in Fig. 2(b). The orbit set $\Omega(s)$ has two blocks: $\{1, 4, 5, 6\}$ and $\{2, 3\}$. Note that $\Omega(t) \prec \Omega(s)$ and $\mathrm{Fix}(t) \supset \mathrm{Fix}(s)$. ■

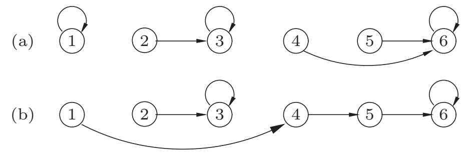

Fig. 2. Non-decreasing transformations $t = [1, 3, 3, 5, 6, 6]$ and $s = [4, 3, 3, 6, 6, 6]$.

Define the transformation $t_{\max} = [2, 3, \ldots, n, n]$. The subscript "max" is chosen because $\Omega(t_{\max}) = \{Q\}$ is the maximal element in the lattice Π_Q. Clearly $t_{\max} \in \mathcal{F}_Q$ and $\mathrm{Fix}(t_{\max}) = \{n\}$. For any submonoid S of $\mathcal{F}_Q$, let $S[t_{\max}]$ be the smallest monoid containing $t_{\max}$ and all elements of S.

Lemma 11. *Let S be a $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$. Then*

(1) *$S[t_{\max}]$ is $\mathcal{J}$-trivial.*
(2) *Let $\mathcal{A} = (Q, \Sigma, \delta, 1, \{n\})$ be the DFA in which each $a \in \Sigma$ defines a distinct transformation in $S[t_{\max}]$. Then $\mathcal{A}$ is minimal.*

Proof. (1) By Theorem 8, it is sufficient to prove that for any $t \in S$, $\Omega(t) \vee \Omega(t_{\max}) = \Omega(tt_{\max})$ and $\Omega(t_{\max}) \vee \Omega(t) = \Omega(t_{\max}t)$. Note that $\Omega(t_{\max}) = \{Q\}$; so we have $\Omega(t) \vee \Omega(t_{\max}) = \Omega(t_{\max}) \vee \Omega(t) = \{Q\}$. On the other hand, since $S \subseteq \mathcal{F}_Q$ and $t_{\max} \in \mathcal{F}_Q$, both $tt_{\max}$ and $t_{\max}t$ are non-decreasing as well. Suppose $j \in \mathrm{Fix}(tt_{\max})$; then $j(tt_{\max}) = (jt)t_{\max} = j$. Since $t_{\max}$ is non-decreasing, $jt \leqslant j$; and since t is also non-decreasing, $j \leqslant jt$. Hence $jt = j$, and $jt_{\max} = j$, which implies that $j \in \mathrm{Fix}(t_{\max})$ and $j = n$. Then $\mathrm{Fix}(tt_{\max}) = \{n\}$ and $\Omega(tt_{\max}) = \{Q\}$. Similarly, $\mathrm{Fix}(t_{\max}t) = \{n\}$ and $\Omega(t_{\max}t) = \{Q\}$. Therefore $S[t_{\max}]$ is also $\mathcal{J}$-trivial.

(2) Suppose $a_0 \in \Sigma$ performs the transformation $t_{\max}$. Each state $p \in Q$ can be reached from the initial state 1 by the word $u = a_0^{p-1}$, and p accepts the word $v = a_0^{n-p}$, while all other states reject v. So $\mathcal{A}$ is minimal. □

For any $\mathcal{J}$-trivial submonoid S of $\mathcal{F}_Q$, we denote by $\mathcal{A}(S, t_{\max})$ the DFA in Lemma 11. Then $\mathcal{A}(S, t_{\max})$ is the quotient DFA of some $\mathcal{J}$-trivial regular language L. Next, we have

Lemma 12. *Let S be a $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$. For any $t, s \in S$, if* $\mathrm{Fix}(t) = \mathrm{Fix}(s)$*, then* $\Omega(t) = \Omega(s)$.

Proof. Pick any $t, s \in S$ such that $\mathrm{Fix}(t) = \mathrm{Fix}(s)$. If $t = s$, then it is trivial that $\Omega(t) = \Omega(s)$. Assume $t \neq s$, and $\Omega(t) \neq \Omega(s)$. By Part 2 of Lemma 9, we have $\Omega(t) \not\preceq \Omega(s)$ and $\Omega(s) \not\preceq \Omega(t)$. Then there exists $i \in Q$ such that $\omega_t(i) \not\subseteq \omega_s(i)$. Let $p = \max(\omega_t(i))$. We define $q \in Q$ as follows. If $\max(\omega_t(i)) \neq \max(\omega_s(i))$, then let $q = \max(\omega_s(i))$; so $q \neq p$. Otherwise $\max(\omega_t(i)) = \max(\omega_s(i))$, and there exists $j \in \omega_t(i)$ such that $j \notin \omega_s(i)$; let $q = \max(\omega_s(j))$. Now $p = \max(\omega_t(j)) = \max(\omega_t(i)) = \max(\omega_s(i))$, and since $j \notin \omega_s(i)$, we have $q \neq p$ as well. Note that

$p, q \in \mathrm{Fix}(t) = \mathrm{Fix}(s)$ in both cases. Consider the DFA $\mathcal{A}(S, t_{\max})$ with alphabet Σ, and suppose that $a \in \Sigma$ performs t and $b \in \Sigma$ performs s. Let $\mathcal{B}$ be the DFA $\mathcal{A}(S, t_{\max})$ restricted to $\{a, b\}$. Since $p \in \omega_t(i)$ and $q \in \omega_s(i)$, p, q are in the same component P of $\mathcal{B}$. However, p and q are two distinct maximal states in P, which contradicts Theorem 7. Therefore $\Omega(t) = \Omega(s)$. □

Example 13. *To illustrate one usage of Lemma* 12, *we consider two non-decreasing transformations* $t = [2, 2, 4, 4]$ *and* $s = [3, 2, 4, 4]$. *They have the same set of fixed points* $\mathrm{Fix}(t) = \mathrm{Fix}(s) = \{2, 4\}$. *However,* $\Omega(t) = \{\{1, 2\}, \{3, 4\}\}$ *and* $\Omega(s) = \{\{2\}, \{1, 3, 4\}\}$. *By Lemma* 12, *$t$ and s cannot appear together in a $\mathcal{J}$-trivial monoid. Indeed, consider any minimal DFA $\mathcal{A}$ having at least two inputs a, b such that a performs t and b performs s. The DFA $\mathcal{B}$ of $\mathcal{A}$ restricted to the alphabet $\{a, b\}$ is shown in Fig.* 3. *There is only one component in $\mathcal{B}$, but there are two maximal states* 2 *and* 4. *By Theorem* 7, *the syntactic monoid of $\mathcal{A}$ is not $\mathcal{J}$-trivial.* ■

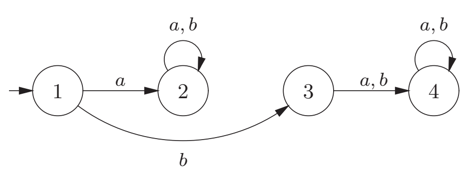

Fig. 3. DFA $\mathcal{B}$ with two inputs a and b, where $t_a = [2, 2, 4, 4]$ and $t_b = [3, 2, 4, 4]$.

Let π be any partition of Q. A block X of π is *trivial* if it contains only one element; otherwise it is *non-trivial.* We define the set $\mathcal{E}(\pi) = \{t \in \mathcal{F}_Q \mid \Omega(t) = \pi\}$. Then

Lemma 14. *If π is a partition of Q with r blocks, where $1 \leqslant r \leqslant n$, then $|\mathcal{E}(\pi)| \leqslant (n - r)!$. Moreover, when $r \neq n$, equality holds if and only if π has exactly one non-trivial block.*

Proof. Suppose $\pi = \{X_1, \dots, X_r\}$, and $|X_i| = k_i$ for each i, $1 \leqslant i \leqslant r$. Without loss of generality, we can rearrange blocks X_i so that $k_1 \leqslant \cdots \leqslant k_r$. Let $t \in \mathcal{E}(\pi)$ be any transformation. Then $t \in \mathcal{F}_Q$, and hence $\mathrm{Fix}(t) = \mathrm{Max}(\Omega(t)) = \mathrm{Max}(\pi)$. Consider each block X_i, and suppose $X_i = \{j_1, \dots, j_{k_i}\}$ with $j_1 < \cdots < j_{k_i}$. Since $j_{k_i} = \max(X_i)$, we have $j_{k_i} \in \mathrm{Fix}(t)$ and $j_{k_i}t = j_{k_i}$. On the other hand, if $1 \leqslant l < k_i$, then $j_l \notin \mathrm{Max}(\pi)$, and since $t \in \mathcal{F}_Q$, we have $j_l t > j_l$; since $j_l t \in \omega_t(j_l) = X_i$, $j_l t \in \{j_{l+1}, \dots, j_{k_i}\}$. So there are $(k_i - 1)!$ different $t|_{X_i}$, and there are $\prod_{i=1}^{r}(k_i - 1)!$ different transformations t in $\mathcal{E}(\pi)$.

Clearly, if $r = 1$, then $k_r = n$ and $|\mathcal{E}(\pi)| = (n - 1)!$. Assume $r \geqslant 2$. Note that $k_i \geqslant 1$ for all i, $1 \leqslant i \leqslant r$, and $\sum_{i=1}^{r} k_i = n$. If $k_1 = \cdots = k_{r-1} = 1$, then $k_r = n - r + 1$, and $|\mathcal{E}(\pi)| = (k_r - 1)! \prod_{i=1}^{r-1} 0! = (n - r)!$. Otherwise, let h be the smallest index such that $k_h > 1$. For all i, $h \leqslant i \leqslant r - 1$, since $k_i \leqslant k_r$, we have

$(k_i-1)! < (k_i-1)^{k_i-1} \leqslant (k_r-1)^{k_i-1}$. Then

$$\begin{aligned}|\mathcal{E}(\pi)| &= (k_r-1)!\prod_{i=1}^{h-1}0!\prod_{i=h}^{r-1}(k_i-1)! < (k_r-1)!\prod_{i=h}^{r-1}(k_r-1)^{k_i-1}\\ &= (k_r-1)!\cdot(k_r-1)^{\sum_{i=h}^{r-1}(k_i-1)}\\ &< (k_r-1)!\cdot k_r(k_r+1)\cdots\left(k_r-1+\sum_{i=h}^{r-1}(k_i-1)\right)\\ &= (k_r-1)!\cdot k_r(k_r+1)\cdots(n-r) = (n-r)!\end{aligned}$$

Therefore the lemma holds. □

Example 15. *Suppose $n = 10$, $r = 3$, and consider the partition $\pi = \{X_1, X_2, X_3\}$, where $X_1 = \{1, 2, 5\}$, $X_2 = \{3, 7\}$, and $X_3 = \{4, 6, 8, 9, 10\}$. Then $k_1 = |X_1| = 3$, $k_2 = |X_2| = 2$, and $k_3 = |X_3| = 5$. Let $t \in \mathcal{E}(\pi)$ be an arbitrary transformation; then $\mathrm{Fix}(t) = \{5, 7, 10\}$. For any $j \in X_1$, if $j = 1$, then jt could be 2 or 5; otherwise $j = 2$ or 5, and jt must be 5. So there are $(k_1-1)! = 2!$ different $t|_{X_1}$. Similarly, there are $(k_2-1)! = 1!$ different $t|_{X_2}$ and $(k_3-1)! = 4!$ different $t|_{X_3}$. So $|\mathcal{E}(\pi)| = 2!1!4! = 48$.*

Consider another partition $\pi' = \{X_1', X_2', X_3'\}$ with three blocks, where $X_1' = \{5\}$, $X_2' = \{7\}$, and $X_3' = \{1, 2, 3, 4, 6, 8, 9, 10\}$. Now $k_1 = |X_1'| = 1$, $k_2 = |X_2'| = 1$, and $k_3 = |X_3'| = 8$. We have $\mathrm{Max}(\pi') = \mathrm{Max}(\pi) = \{5, 7, 10\}$. Then, for any $t \in \mathcal{E}(\pi')$, $\mathrm{Fix}(t) = \{5, 7, 10\}$ as well. Since $k_1 = k_2 = 1$, both $t|_{X_1}$ and $t|_{X_2}$ are unique. There are $(k_3-1)! = 7!$ different $t|_{X_3}$. Together we have $|\mathcal{E}(\pi')| = 1!1!7! = (10-3)! = 5040$, which is the upper bound in Lemma 14 for $n = 10$ and $r = 3$. ■

Note that, for any $t \in \mathcal{F}_Q$, we have $n \in \mathrm{Fix}(t)$. Let $\mathcal{P}_n(Q)$ be the set of all subsets Z of Q such that $n \in Z$. Then we obtain the following upper bound.

Proposition 16. *For $n \geqslant 1$, if S is a $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$, then*

$$|S| \leqslant \sum_{r=1}^{n}\binom{n-1}{r-1}(n-r)! = \lfloor e(n-1)!\rfloor.$$

Proof. Assume S is a $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$. For any $Z \in \mathcal{P}_n(Q)$, let $S_Z = \{t \in S \mid \mathrm{Fix}(t) = Z\}$. Then $S = \bigcup_{Z\in\mathcal{P}_n(Q)} S_Z$, and for any $Z_1, Z_2 \in \mathcal{P}_n(Q)$ with $Z_1 \neq Z_2$, $S_{Z_1} \cap S_{Z_2} = \emptyset$.

Pick any $Z \in \mathcal{P}_n(Q)$. By Lemma 12, for any $t, s \in S_Z$, since $\mathrm{Fix}(t) = \mathrm{Fix}(s) = Z$, we have $\Omega(t) = \Omega(s) = \pi$ for some partition $\pi \in \Pi_Q$. Then $S_Z \subseteq \mathcal{E}(\pi)$. Suppose $r = |Z|$. By Lemma 14, $|S_Z| \leqslant |\mathcal{E}(\pi)| \leqslant (n-r)!$. Since $n \in Z$, $1 \leqslant r \leqslant n$; and since there are $\binom{n-1}{r-1}$ different $Z \in \mathcal{P}_n(Q)$, we have

$$|S| = \sum_{Z\in\mathcal{P}_n(Q)} |S_Z| \leqslant \sum_{r=1}^{n}\binom{n-1}{r-1}(n-r)! = \sum_{r=1}^{n}\frac{(n-1)!}{(r-1)!} = \lfloor e(n-1)!\rfloor.$$

The last equality is a well-known identity in combinatorics [15]. □

The above upper bound is met by the following monoid $\mathcal{S}_n$. For any $Z \in \mathcal{P}_n(Q)$, suppose $Z = \{j_1, \dots, j_r, n\}$ such that $j_1 < \cdots < j_r < n$ for some $r \geqslant 0$; then we define partition $\pi_Z = \{Q\}$ if $Z = \{n\}$, and $\pi_Z = \{\{j_1\}, \dots, \{j_r\}, Q \setminus \{j_1, \dots, j_r\}\}$ otherwise. Let

$$\mathcal{S}_n = \bigcup_{Z \in \mathcal{P}_n(Q)} \mathcal{E}(\pi_Z).$$

Example 17. *Suppose $n = 4$; then $|\mathcal{P}_4(Q)| = 2^3 = 8$. First consider $Z = \{1, 3, 4\} \in \mathcal{P}_4(Q)$. By definition, $\pi_Z = \{\{1\}, \{3\}, \{2, 4\}\}$. There is only one transformation $t_1 = [1, 4, 3, 4]$ in $\mathcal{E}(\pi_Z)$. If $Z' = \{3, 4\}$, then $\pi_{Z'} = \{\{3\}, \{1, 2, 4\}$. There are two transformations $t_2 = [2, 4, 3, 4]$ and $t_3 = [4, 4, 3, 4]$ in $\mathcal{E}(\pi_{Z'})$. Table 1 summarizes the number of transformations in $\mathcal{E}(\pi_Z)$ for each $Z \in \mathcal{P}_4(Q)$. Note that the set $\mathcal{S}_4$ contains 16 transformations in total.* ■

Table 1. Number of transformations in $\mathcal{E}(\pi_Z)$ for each $Z \in \mathcal{P}_4(Q)$.

Z	Blocks of π_Z	$\lvert\mathcal{E}(\pi_Z)\rvert$
{1,2,3,4}	{1}, {2}, {3}, {4}	1
{1,2,4}	{1}, {2}, {3,4}	1
{1,3,4}	{1}, {3}, {2,4}	1
{2,3,4}	{2}, {3}, {1,4}	1
{1,4}	{1}, {2,3,4}	2
{2,4}	{2}, {1,3,4}	2
{3,4}	{3}, {1,2,4}	2
{4}	{1,2,3,4}	6

Proposition 18. *For $n \geqslant 1$, the set $\mathcal{S}_n$ is a $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$ with cardinality*

$$|\mathcal{S}_n| = \sum_{r=1}^{n} \binom{n-1}{r-1} (n-r)! = \lfloor e(n-1)! \rfloor.$$

Proof. First we prove the following claim:

Claim: For any $t, s \in \mathcal{S}_n$, $\Omega(ts) = \pi_Z$ for some $Z \in \mathcal{P}_n(Q)$.

Let $t \in \mathcal{E}(\pi_{Z_1})$ and $s \in \mathcal{E}(\pi_{Z_2})$ for some $Z_1, Z_2 \in \mathcal{P}_n(Q)$. Suppose $\Omega(ts) \neq \pi_Z$ for any $Z \in \mathcal{P}_n(Q)$. Then there exists a block $X_0 \in \Omega(ts)$ such that $n \notin X_0$ and $|X_0| \geqslant 2$. Let $h = \max(X_0)$; then $h(ts) = h$, and since both t and s are non-decreasing, $ht = h$ and $hs = h$. Since $h \neq n$, both $\omega_t(h)$ and $\omega_s(h)$ are trivial blocks. Now let $j \in X_0$ such that $j(ts) = h$ and $j \neq h$. If $jt \neq h$, then $jt \in \omega_s(h)$ and $\omega_s(h)$ is a non-trivial block, a contradiction. Otherwise $jt = h$, then $\omega_t(h)$ is a non-trivial block, a contradiction again. So the claim holds.

By the claim, for any $t, s \in \mathcal{S}_n$, since $\Omega(ts) = \pi_Z$ for some $Z \in \mathcal{P}_n(Q)$, $ts \in \mathcal{E}(\pi_Z) \subseteq \mathcal{S}_n$. Hence $\mathcal{S}_n$ is a submonoid of $\mathcal{F}_Q$.

Next we show that $\mathcal{S}_n$ is $\mathcal{J}$-trivial. Pick any $t, s \in \mathcal{S}_n$, and suppose $t \in \mathcal{E}(\pi_{Z_1})$ and $s \in \mathcal{E}(\pi_{Z_2})$ for some $Z_1, Z_2 \in \mathcal{P}_n(Q)$. Suppose $\mathrm{Max}(Z_1) \cap \mathrm{Max}(Z_2) = \{j_1, \ldots, j_r, n\}$, for some $r \geqslant 0$. Then we have $Z_1 \vee Z_2 = \{\{j_1\}, \ldots, \{j_r\}, X\}$, where $X = Q \setminus \{j_1, \ldots, j_r\}$ and $n \in X$. On the other hand, by the claim, $\Omega(ts) = \{\{p_1\}, \ldots, \{p_k\}, Y\}$ for some $p_1, \ldots, p_k \in Q$, where $Y = Q \setminus \{p_1, \ldots, p_k\}$ and $n \in Y$. Note that, since $\mathcal{S}_n \subseteq \mathcal{F}_Q$, $\mathrm{Max}(\Omega(ts)) = \mathrm{Fix}(ts) = \mathrm{Fix}(t) \cap \mathrm{Fix}(s) = \mathrm{Max}(Z_1) \cap \mathrm{Max}(Z_2)$. Then $r = k$ and $\{j_1, \ldots, j_r\} = \{p_1, \ldots, p_k\}$. Hence $\Omega(t) \vee \Omega(s) = Z_1 \vee Z_2 = \Omega(ts)$. By Theorem 8, $\mathcal{S}_n$ is $\mathcal{J}$-trivial.

For any $Z \in \mathcal{P}_n(Q)$ with $|Z| = r$, where $1 \leqslant r \leqslant n$, we have $\pi_Z = \{X_1, \ldots, X_r\}$ with $k_i = |X_i| = 1$ for $1 \leqslant i < r$, and $k_r = |X_r|$. By Lemma 14, $|\mathcal{E}(\pi_Z)| = (n-r)!$. Moreover, if $Z_1 \neq Z_2$, then $\mathcal{E}(\pi_{Z_1}) \cap \mathcal{E}(\pi_{Z_2}) = \emptyset$. Since $n \in Z$ is fixed, there are $\binom{n-1}{r-1}$ different Z. Therefore $|\mathcal{S}_n| = \sum_{r=1}^{n} \binom{n-1}{r-1}(n-r)! = \lfloor e(n-1)! \rfloor$. □

We now define a generating set of the monoid $\mathcal{S}_n$. Suppose $n \geqslant 1$. For any $Z \in \mathcal{P}_n(Q)$, if $Z = Q$, then let $t_Z = \mathbf{1}$. Otherwise, let $h_Z = \max(Q \setminus Z)$, and let t_Z be a transformation of Q defined by: For all $j \in Q$,

$$jt_Z \stackrel{\text{def}}{=} \begin{cases} j & \text{if } j \in Z, \\ n & \text{if } j = h_Z, \\ h_Z & \text{otherwise.} \end{cases}$$

Let $\mathcal{GS}_n = \{t_Z \mid Z \in \mathcal{P}_n(Q)\}$.

Proposition 19. *For $n \geqslant 1$, the monoid $\mathcal{S}_n$ can be generated by the set $\mathcal{GS}_n$ of 2^{n-1} transformations of Q.*

Proof. First, for any $t_Z \in \mathcal{GS}_n$, where $Z \in \mathcal{P}_n(Q)$, we have $\Omega(t_Z) = \pi_Z$; hence $t_Z \in \mathcal{E}(\pi_Z) \subseteq \mathcal{S}_n$. So $\mathcal{GS}_n \subseteq \mathcal{S}_n$ and $\langle \mathcal{GS}_n \rangle \subseteq \mathcal{S}_n$, where $\langle \mathcal{GS}_n \rangle$ denotes the semigroup generated by $\mathcal{GS}_n$.

Fix arbitrary $Z \in \mathcal{P}_n(Q)$, and suppose $U = Q \setminus Z$. If $U = \emptyset$, then $\pi_Z = \{\{1\}, \ldots, \{n\}\}$ and $\mathcal{E}(\pi_Z) = \{\mathbf{1}\} \subseteq \langle \mathcal{GS}_n \rangle$. Assume $U \neq \emptyset$ in the following. Let Y be the only non-trivial block in π_Z. Note that $Y = U \cup \{n\}$ and $h_Z = \max(U)$. For any $t \in \mathcal{E}(\pi_Z)$, since $\mathrm{Fix}(t) = Z$ and $h_Z \notin Z$, $h_Z t > h_Z$; and since Y is an orbit of t, $h_Z t = n$. We prove by induction on $|U| = |Q \setminus Z|$ that $\mathcal{E}(\pi_Z) \subseteq \langle \mathcal{GS}_n \rangle$.

(1) If $U = \{h_Z\}$, then $Y = \{h_Z, n\}$. So $t = (h_Z \to n) = t_Z \subseteq \langle \mathcal{GS}_n \rangle$.
(2) Otherwise $U = \{h_1, \ldots, h_l, h_Z\}$ for some $h_1 < \cdots < h_l < h_Z < n$ and $l \geqslant 1$. Assume that, for any $Z' \in \mathcal{P}_n(Q)$ with $|Q \setminus Z'| \leqslant l$, we have $\mathcal{E}(\pi_{Z'}) \subseteq \langle \mathcal{GS}_n \rangle$. Then $Y = \{h_1, \ldots, h_l, h_Z, n\}$, and $t_Z = (h_Z \to n)(h_l \to h_Z) \cdots (h_1 \to h_Z)$. For any $t \in \mathcal{E}(\pi_Z)$, since Y is an orbit of t and $Q \setminus Y \subseteq \mathrm{Fix}(t)$, t must have the form $t = (h_Z \to n)(h_l \to j_l) \cdots (h_1 \to j_1)$, where $j_i \in \{h_{i+1}, \ldots, h_l, h_Z, n\}$ for $i = 1, \ldots, l$. Let $\{h_1, \ldots, h_l\} = V \cup W$ such that $h_i \in V$ if and only if $j_i = h_i t = h_Z$. Suppose $V = \{h_{p_1}, \ldots, h_{p_k}\}$ and $W = \{h_{q_1}, \ldots, h_{q_m}\}$, where $h_{p_1} < \cdots < h_{p_k}$,

$h_{q_1} < \cdots < h_{q_m}$, $0 \leqslant k, m \leqslant l$ and $l = k + m$. Let $t_1 = (h_Z \to n)$, $t_2 = (h_Z \to n)(h_{p_1} \to h_Z) \cdots (h_{p_k} \to h_Z)$, and $t_3 = (h_{p_1} \to n) \cdots (h_{p_k} \to n)(h_{q_m} \to j_{q_m}) \cdots (h_{q_1} \to j_{q_1})$. Note that $t_1 = t_{Z'}$ for $Z' = Q \setminus \{h_Z\}$, and $t_2 = t_{Z''}$ for $Z'' = Q \setminus \{h_{p_1}, \ldots, h_{p_k}, h_Z\}$. Also note that $\mathrm{Fix}(t_3) = \mathrm{Fix}(t) \cup \{h_Z\}$, and since $j_{q_i} = h_{q_i} t \in U \setminus \{h_Z\}$ for all $h_{q_i} \in W$, we have $t_3 \in \mathcal{E}(\pi_{Z'''})$ for $Z''' = Z \cup \{h_Z\}$. By assumption, $t_3 \in \langle \mathcal{GS}_n \rangle$. Now

$$\begin{aligned} t_1 t_2 t_3 &= (h_Z \to n) \circ (h_Z \to n)(h_{p_1} \to h_Z) \cdots (h_{p_k} \to h_Z) \\ &\quad \circ (h_{p_1} \to n) \cdots (h_{p_k} \to n)(h_{q_m} \to j_{q_m}) \cdots (h_{q_1} \to j_{q_1}) \\ &= (h_Z \to n)(h_{p_1} \to h_Z) \cdots (h_{p_k} \to h_Z)(h_{q_m} \to j_{q_m}) \cdots (h_{q_1} \to j_{q_1}) \\ &= t. \end{aligned}$$

Thus $t \in \langle \mathcal{GS}_n \rangle$ and $\mathcal{E}(\pi_Z) \subseteq \langle \mathcal{GS}_n \rangle$.

By induction, $\mathcal{S}_n = \bigcup_{Z \in \mathcal{P}_n(Q)} \mathcal{E}(\pi_Z) \subseteq \langle \mathcal{GS}_n \rangle$. Therefore $\mathcal{S}_n = \langle \mathcal{GS}_n \rangle$. Since there are 2^{n-1} different $Z \in \mathcal{P}_n(Q)$, there are 2^{n-1} transformations in $\mathcal{GS}_n$. □

Example 20. *Suppose $n = 5$. Consider $Z = \{3, 5\} \in \mathcal{P}_5(Q)$, and $t = [2, 4, 3, 5, 5] \in \mathcal{E}(\pi_Z)$. The transition graph of t is shown in Fig.* 4(*a*). *As in Proposition* 19*, we have $U = \{1, 2, 4\}$ and $h_Z = 4$. To show that $t \in \langle \mathcal{GS}_5 \rangle$, we find $V = \{2\}$ and $W = \{1\}$. Then let $t_1 = (4 \to 5)$, $t_2 = (4 \to 5)(2 \to 4)$, and $t_3 = (2 \to 5)(1 \to 2)$. We assume that $t_3 \in \langle \mathcal{GS}_5 \rangle$; in fact, $t_3 = t_{Z'''}$ for $Z''' = \{3, 4, 5\}$ in this example. The transition graphs of t_1, t_2, and t_3 are shown in Figs.* 4(*b*), 4(*c*), *and* 4(*d*), *respectively. One can verify that $t = t_1 t_2 t_3$, and hence $t \in \langle \mathcal{GS}_5 \rangle$.* ■

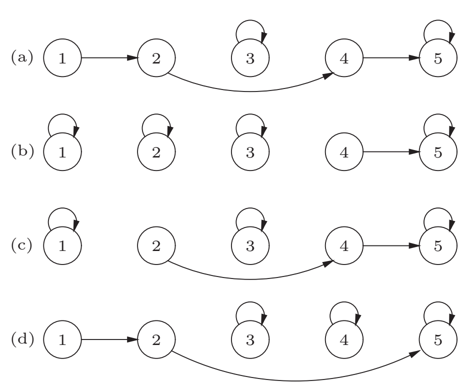

Fig. 4. Transition graphs of $t = [2, 4, 3, 5, 5]$, $t' = [1, 4, 3, 5, 5]$, and $t_{Z''} = [2, 5, 3, 4, 5]$.

Now, by Propositions 16, 18, and 19, we have

Theorem 21. *Let $L \subseteq \Sigma^*$ be a $\mathcal{J}$-trivial regular language with quotient complexity $n \geqslant 1$. Then its syntactic complexity $\sigma(L)$ satisfies $\sigma(L) \leqslant \lfloor e(n-1)! \rfloor$, and this bound is tight if $|\Sigma| \geqslant 2^{n-1}$.*

It was shown by Saito [16] that, if S is a $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$, then $\Omega(S) = \{\Omega(t) \mid t \in S\} \subseteq \Pi_Q$ forms a $\vee$-semilattice, called a $\mathcal{J}$-$\vee$-*semilattice*, such that $\mathrm{Max}(\Omega(t) \vee \Omega(s)) = \mathrm{Fix}(t) \cap \mathrm{Fix}(s)$. Let $\mathcal{P}_\vee(\Pi_Q)$ be the set of all $\mathcal{J}$-$\vee$-semilattices that are subsets of Π_Q. A maximal $\mathcal{J}$-trivial submonoid S of $\mathcal{F}_Q$ corresponds to a maximal element P in $\mathcal{P}_\vee(\Pi_Q)$, with respect to set inclusion, such that $S = \bigcup_{\pi \in P} \mathcal{E}(\pi)$. $P \in \mathcal{P}_\vee(\Pi_Q)$ is called *full* if $\{\mathrm{Max}(\pi) \mid \pi \in P\} = \mathcal{P}_n(Q)$, which is an maximal element in $\mathcal{P}_\vee(\Pi_Q)$ with respect to set inclusion. The monoid $\mathcal{S}_n$ then corresponds to a full $\mathcal{J}$-$\vee$-semilattice, and hence it is maximal. Saito described all maximal $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$ and those corresponding to full $\mathcal{J}$-$\vee$-semilattices. However, here we consider the $\mathcal{J}$-trivial submonoid of $\mathcal{F}_Q$ with maximum cardinality.

5. Conclusion

We proved that $n!$ and $\lfloor e(n-1)! \rfloor$ are the tight upper bounds on the syntactic complexities of $\mathcal{R}$- and $\mathcal{J}$-trivial languages with n quotients, respectively. For $n \geqslant 2$, the upper bound for $\mathcal{R}$-trivial languages can be met using $1 + \binom{n}{2}$ letters, and the upper bound for $\mathcal{J}$-trivial languages, using 2^{n-1} letters. It remains open whether the upper bound for $\mathcal{J}$-trivial languages can be met with fewer than 2^{n-1} letters. The syntactic complexity of $\mathcal{L}$-trivial languages is also open.

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