COMPLEXITY OF ATOMS OF REGULAR LANGUAGES

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The quotient complexity of a regular language $L$, which is the same as its state complexity, is the number of left quotients of $L$. An atom of a non-empty regular language $L$ with $n$ quotients is a non-empty intersection of the $n$ quotients, which can be uncomplemented or complemented. An NFA is atomic if the right language of every state is a union of atoms. We characterize all reduced atomic NFAs of a given language, i.e., those NFAs that have no equivalent states. We prove that, for any language $L$ with quotient complexity $n$, the quotient complexity of any atom of $L$ with $r$ complemented quotients has an upper bound of $2^n - 1$ if $r = 0$ or $r = n$; for $1 \leq r \leq n - 1$ the bound is $1 + \sum_{k=1}^{r} \sum_{h=k+1}^{n} \binom{n}{h} \binom{h}{k}$.

For each $n \geq 2$, we exhibit a language with $2^n$ atoms which meet these bounds.

Keywords: Atoms; finite automaton; atomic NFA; quotient complexity; regular language; state complexity; syntactic semigroup; witness.

1. Terminology and Notation

In this section we provide some background information, introduce atoms of regular languages, and state our reasons for studying them. For basic properties of regular languages and finite automata see [7, 9].

If $\Sigma$ is a non-empty finite alphabet, then $\Sigma^*$ is the free monoid generated by $\Sigma$. A word is any element of $\Sigma^*$, and the empty word is $\epsilon$. A language over $\Sigma$ is any subset of $\Sigma^*$. The reverse of a language $L$ is denoted by $L^R$ and defined as $L^R = \{w^R \mid w \in L\}$, where $w^R$ is $w$ spelled backwards.

The (left) quotient of a regular language $L$ over an alphabet $\Sigma$ by a word $w \in \Sigma^*$ is the language $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$. It is well known that the quotient of
a regular language is itself regular, and that a language is regular if and only if it has a finite number of distinct quotients. Also, $L$ is its own quotient by the empty word $\varepsilon$, that is $\varepsilon^{-1}L = L$. Note too that the quotient by $u \in \Sigma^*$ of the quotient by $w \in \Sigma^*$ of $L$ is the quotient by $wu$ of $L$, that is, $u^{-1}(w^{-1}L) = (wu)^{-1}L$.

Although quotients have been known for over half a century, atoms were introduced only in 2011 by Brzozowski and Tamm [4]; here we use a slightly different definition for reasons explained at the end of Sec. 2. An atom of a regular language $L$ with quotients $K_0, \ldots, K_{n-1}$ is any non-empty language of the form $\tilde{K}_0 \cap \cdots \cap \tilde{K}_{n-1}$, where $\tilde{K}_i$ is either $K_i$ or $\overline{K_i}$; $\overline{K_i}$ is the complement of $K_i$ with respect to $\Sigma^*$. The atoms of any regular language $L$ have the following properties, which have either been shown in [4] or are easily verified for the new definition:

1. Atoms are regular because quotients are regular and regularity is preserved under complement and intersection.
2. If $L$ has $n$ quotients, it has at most $2^n$ atoms, by the definition of atoms.
3. Atoms are pairwise disjoint because, if two intersections differ, there must be a quotient that is complemented in one intersection but not in the other.
4. The atoms of $L$ partition $\Sigma^*$, since the union of all the intersections is $\Sigma^*$.
5. Every quotient $K$ of $L$ is a (possibly empty) union of atoms, namely all those atoms in which $K$ is not complemented.
6. Every quotient of an atom is a (possibly empty) union of atoms, because the quotient of an intersection of quotients of $L$ is an intersection of quotients of $L$.
7. The complement of $L$ is a union of atoms of $L$, namely all those atoms in which $L$ is complemented.

In summary, the atoms of a regular language are its basic building blocks. A nondeterministic finite automaton (NFA) is a quintuple $\mathfrak{A} = (Q, \Sigma, \eta, I, F)$, where $Q$ is a finite, non-empty set of states, $\Sigma$ is a finite non-empty alphabet, $\eta : Q \times \Sigma \rightarrow 2^Q$ is the transition function, $I \subseteq Q$ is the set of initial states, and $F \subseteq Q$ is the set of final states. As usual, we extend the transition function to functions $\eta' : Q \times \Sigma^* \rightarrow 2^Q$, and $\eta'' : 2^Q \times \Sigma^* \rightarrow 2^Q$, but we use $\eta$ for all three functions.

The language accepted by an NFA $\mathfrak{A}$ is $L(\mathfrak{A}) = \{w \in \Sigma^* | \eta(I, w) \cap F \neq \emptyset\}$. Two NFAs are equivalent if they accept the same language. The right language of a state $q$ of $\mathfrak{A}$ is $L_{q,F}(\mathfrak{A}) = \{w \in \Sigma^* | \eta(q, w) \cap F \neq \emptyset\}$. The right language of a set $S$ of states of $\mathfrak{A}$ is $L_{S,F}(\mathfrak{A}) = \bigcup_{q \in S} L_{q,F}(\mathfrak{A})$; hence $L(\mathfrak{A}) = L_{I,F}(\mathfrak{A})$. A state is empty if its right language is empty. Two states of an NFA are equivalent if their right languages are equal. The left language of a state $q$ of $\mathfrak{A}$ is $L_{I,q} = \{w \in \Sigma^* | q \in \eta(I, w)\}$. A state is unreachable if its left language is empty. An NFA is trim if it has no empty or unreachable states. An NFA is reduced if it has no equivalent states. An NFA is minimal if it has the minimal number of states among all the equivalent NFAs.

A deterministic finite automaton (DFA) is a quintuple $\mathfrak{D} = (Q, \Sigma, \delta, q_0, F)$, where $Q$, $\Sigma$, and $F$ are as in an NFA, $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, and $q_0 \in Q$ is the initial state. It is evident that a DFA is a special type of NFA.
A DFA is minimal if all of its states are reachable, and no two states are equivalent. It is well-known that for every regular language $L$ there exists a unique (up to isomorphism) minimal DFA.

We use the following operations on automata:

1. The determinization operation $D$ applied to an NFA $\mathfrak{A}$ yields a DFA $\mathfrak{A}^D$ obtained by the subset construction, where only subsets reachable from the initial subset of $\mathfrak{A}^D$ are used and the empty subset, if present, is included.

2. The reversal operation $R$ applied to an NFA $\mathfrak{A}$ yields an NFA $\mathfrak{A}^R$, where sets of initial and final states of $\mathfrak{A}$ are interchanged and each transition is reversed.

2. Quotient DFAs and Atomata

Let $L$ be any non-empty regular language, and let its set of quotients be $K = \{K_0, \ldots, K_{n-1}\}$. The quotient of $\varepsilon^{-1}L = L$ is called initial and is denoted by $K_{in}$. The set of final quotients is $F = \{K_i \mid \varepsilon \in K_i\}$. In the following definition we use a 1-1 correspondence $K_i \leftrightarrow K_i$ between quotients $K_i$ of a language $L$ and states $K_i$ of the quotient DFA $\mathfrak{D}$ defined below. We refer to the $K_i$ as quotient symbols.

**Definition 1.** The quotient DFA of $L$ is $\mathfrak{D} = (K, \Sigma, \delta, K_{in}, F)$, where $K = \{K_0, \ldots, K_{n-1}\}$, $\delta(K_i, a) = K_j$ if and only if $a^{-1}K_i = K_j$ for all $K_i, K_j \in K$ and $a \in \Sigma$, $K_{in}$ corresponds to $K_{in}$, and $F = \{K_i \mid K_i \in F\}$.

The quotient DFA has the following properties:

1. The right language of state $K_i$ is the quotient $K_i$.
2. The left language of state $K_i$ is $\{w \in \Sigma^* \mid w^{-1}L = K_i\}$.
3. $L(\mathfrak{D})$ is the right language of state $K_{in}$, and hence $L(\mathfrak{D}) = L$.
4. $\mathfrak{D}$ is minimal, since all the quotients in $K$ are distinct.
5. The complement $\overline{L}$ of $L$ is accepted by the DFA $\mathfrak{D}' = (K, \Sigma, \delta, K_{in}, K \setminus F)$, obtained from $\mathfrak{D}$ by changing the final states.

In summary, the quotients of a regular language define its minimal DFA.

Next, we use atoms instead of quotients as states of an automaton. An atom is initial if it has $L$ (rather than $\overline{L}$) as a term; it is final if it contains $\varepsilon$. Since $L$ is non-empty, it has at least one quotient containing $\varepsilon$. Hence it has exactly one final atom, the atom $K_0 \cap \cdots \cap K_{n-1}$, where $K_i = K_j$ if $\varepsilon \in K_i$, and $K_i = K_j$ otherwise. If the intersection $K_0 \cap \cdots \cap K_{n-1}$ is non-empty, then we call it the negative atom; all the other atoms are positive. Let the number of atoms be $m$ and let the number of positive atoms be $p$. Let $A = \{A_0, \ldots, A_{m-1}\}$ be the set of atoms of $L$. By convention, $\emptyset$ is the set of initial atoms, $A_{p-1}$ is the final atom, and the negative atom, if present, is $A_{m-1}$. The negative atom can never be final, since there must be at least one complemented final quotient in its intersection.

As above, we use a 1-1 correspondence $A_i \leftrightarrow A_i$ between atoms $A_i$ of a language $L$ and the states $A_i$ of the NFA $\mathfrak{A}$ defined below. We refer to the $A_i$ as atom symbols.
Definition 2. The atomaton of $L$ is the NFA $\mathfrak{A} = (A, \Sigma, \alpha, A_1, \{A_{p−1}\})$, where $A = \{A_i \mid A_i \in A\}$, $A_j \in \alpha(A_i, a)$ if and only if $aA_j \subseteq A_i$, for all $A_i, A_j \in A$ and $a \in \Sigma$, $A_f = \{A_i \mid A_i \in \mathcal{I}\}$, and $A_{p−1}$ corresponds to $A_{p−1}$.

The atomaton of any regular language $L$ has the properties listed below; these results are either proved in [4], or easily verified for the new definition of atomaton.

1. The right language of state $A_i$ is the atom $A_i$.
2. If $A_i$ is not the negative atom, the left language of state $A_i$ is $L_{A_i, A_i}(\mathfrak{A}) = ((x_i)^{-1}L)^R$, for $i \in \{0, \ldots, p − 1\}$ and $x \in A_i$, and this left language is non-empty. The left language of the negative atom is empty.
3. The language accepted by the atomaton is $L(\mathfrak{A}) = L_{A_i, A_{p−1}} = L$.
4. $\mathfrak{A}$ is reduced, since all the atoms in $A$ are distinct.
5. The complement $\mathcal{C}$ of $L$ is accepted by $\mathfrak{A}' = (A, \Sigma, \alpha, A'_1, \{A_{p−1}\})$, obtained from $\mathfrak{A}$ by changing the initial states to states whose atoms have $\mathcal{C}$ as a term.
6. The determinized version $\mathfrak{A}^D$ of $\mathfrak{A}$ is isomorphic to the quotient DFA of $L$.
7. The reverse $\mathfrak{A}^R$ of $\mathfrak{A}$ is isomorphic to the quotient DFA of $L^R$.

In summary, the atoms of a regular language define a unique reduced NFA of the language, and this NFA has some remarkable properties.

Example 3. Let $L_2 \subseteq [a, c]^*$ be defined by the quotient equations below (left) and recognized by the DFA $\mathcal{D}_2$ of Fig. 1(a), where the initial state is indicated by an arrow and the final state, by a double circle.

$$K_0 = aK_1 \cup cK_0, \quad K_0 \cap K_1 = a(K_0 \cap K_1) \cup \{[K_0 \cap K_1] \cup (K_0 \cap K_1)]$$,

$$K_1 = aK_0 \cup cK_0 \cup \epsilon, \quad K_0 \cap \overline{K_1} = a(\overline{K_0} \cap K_1),$$

$$K_0 \cap K_1 = a(K_0 \cap K_1) \cup \epsilon,$$

$$\overline{K_0} \cap \overline{K_1} = a(\overline{K_0} \cap \overline{K_1}) \cup \epsilon$$

$K_0 \cap K_1 = (aK_1 \cup cK_0) \cap (aK_0 \cup cK_0 \cup \epsilon)$

$\overline{K_0} \cap \overline{K_1} = (aK_0 \cup cK_0) \cap (a\overline{K_0} \cup c\overline{K_0})$

$\overline{K_0} \cap \overline{K_1} = a(\overline{K_0} \cap \overline{K_1}) = a(\overline{K_0} \cap K_1)$.

The atomaton $\mathfrak{A}_2$ is in Fig. 1(b); here each atom is denoted by $A_P$, where $P$ is the set of uncomplemented quotients. Thus $K_0 \cap \overline{K_1}$ becomes $A_{(0)}$, etc., and we represent the sets in the subscripts without brackets and commas. The reverse $\mathcal{D}_2^R$ of $\mathcal{D}_2$ is in Fig. 1(c). The determinized reverse $\mathcal{D}_2^{RD}$ is in Fig. 1(d); this is the minimal DFA for $L_2^R$, the reverse of $L_2$. The reverse $\mathfrak{A}_2^R$ of the atomaton is in Fig. 1(e). Note that $\mathcal{D}_2^{RD}$ and $\mathfrak{A}_2^R$ are isomorphic.

We are now in a position to explain the differences between our present definition of an atom and that of [4]. The definition in [4] did not consider the intersection
of all the complemented quotients to be an atom, and so all atoms were positive. It was shown in [4] that the reverse of the atomaton with only positive atoms is the trim version of the minimal DFA of $L_R$. With the negative atom, we avoid the trimming operation; so the reverse of the atomaton is the minimal DFA of $L_R$. Also, with the negative atom, a language $L$ and its complement language $\overline{L}$ have the same atoms. In addition, we have symmetry between the atoms with 0 and $n$ complemented quotients, and the same upper bounds on quotient complexity for both, as will be shown in Sec. 5.

One might also consider a model in which there is an empty atom. Then there would be unnecessary transitions from every atom under every input to the empty atom. If this atomaton were reversed, the DFA for $L_R$ would have an unreachable state. For this reason we avoided this definition.

3. Atomic NFAs

We show now that atoms lead naturally to a new class of NFAs: DFAs and atomata are special cases of atomic NFAs introduced in [4] and studied further in [5].

In this section we deal only with trim NFAs; thus we do not include the negative atom in the atomaton, if present. This also implies that we do not include the empty state when we determinize.

**Definition 4.** An NFA $\mathcal{R} = (Q, \Sigma, \eta, I, F)$ is atomic if for every $q \in Q$, the right language $L_{q,F}(\mathcal{R})$ of $q$ is a union of some atoms of $L(\mathcal{R})$.

The following theorem, slightly restated, was proved in [4]:

**Theorem 5 (Atomicity)** A trim NFA $\mathcal{R}$ is atomic if and only if $\mathcal{R}^{RD}$ is minimal.

This theorem allows us to test whether an NFA $\mathcal{R}$ accepting a language $L$ is atomic. To do this, reverse $\mathcal{R}$ and apply the subset construction. Then $\mathcal{R}$ is atomic if and only if $\mathcal{R}^{RD}$ is isomorphic to the minimal DFA of $L_R$. 

**Fig. 1.** (a) DFA $\mathcal{D}_2$; (b) atomaton $\mathcal{A}_2$; (c) NFA $\mathcal{D}_2^R$; (d) DFA $\mathcal{D}_2^{RD}$; and (e) DFA $\mathcal{A}_2^R$. 

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If we allow equivalent states, there is an infinite number of atomic NFAs, but their behaviours are not distinct; hence we consider only reduced NFAs. Suppose $\mathcal{B} = (\mathcal{B}, \Sigma, \beta, B_1, B_F)$ is any trim reduced atomic NFA accepting $L$. Since $\mathcal{B}$ is atomic, the right language of any state of $\mathcal{B}$ is a union of positive atoms of $L$; hence the states of $\mathcal{B}$ can be represented by sets of positive atom symbols. Because $\mathcal{B}$ is reduced, no set of atom symbols appears twice. Thus the state set $\mathcal{B}$ is a collection of non-empty sets of positive atom symbols.

**Theorem 6 (Legality)** Suppose $L$ is a regular language, its atomaton is $A = (A, \Sigma, \alpha, A_1, \{A_p-1\})$, and $\mathcal{B} = (B, \Sigma, \beta, B_1, B_F)$ is a trim NFA, where $B = \{B_1, \ldots, B_s\}$ is a collection of sets of positive atom symbols and $B_1, B_F \subseteq B$.

If $B_k \subseteq B$, define $S(B_k) = \bigcup_{B_i \in B_k} B_i$ to be the set of atom symbols appearing in the sets $B_i$ of $B_k$. Then $\mathcal{B}$ is a reduced atomic NFA of $L$ if and only if it satisfies the following conditions:

1. $S(B_1) = A_1$.
2. For all $B_i \in B$, $S(\beta(B_i, a)) = \alpha(B_i, a)$.
3. For all $B_i \in B$, we have $B_i \in B_F$ if and only if $A_{p-1} \in B_i$.

Before proving the theorem, we require the following lemma:

**Lemma 7.** If $\mathcal{B}$ satisfies Condition 2 of Theorem 6, then $S(\beta(B_i, w)) = \alpha(B_i, w)$ for every $B_i \in B$ and $w \in \Sigma^*$.

**Proof.** For $w = \varepsilon$, we have $S(\beta(B_i, \varepsilon)) = S(B_i) = B_i$, and $\alpha(B_i, \varepsilon) = B_i$; so the claim holds for this case.

Assume that $S(\beta(B_i, w)) = \alpha(B_i, w)$ for all $B_i \in B$ and all $w \in \Sigma^*$ with length less than or equal to $l \geq 0$. We prove that $S(\beta(B_i, wa)) = \alpha(B_i, wa)$ for every $a \in \Sigma$. Let $\beta(B_i, w) = \{B_{i_1}, \ldots, B_{i_n}\}$ for some $B_{i_1}, \ldots, B_{i_n} \in B$. Since $\beta(B_i, wa) = \beta(\beta(B_i, w), a) = \beta(B_i, a) \cup \cdots \cup \beta(B_{i_n}, a)$, we have $S(\beta(B_i, wa)) = S(\beta(B_{i_1}, a) \cup \cdots \cup \beta(B_{i_n}, a)) = S(\beta(B_{i_1}, a)) \cup \cdots \cup S(\beta(B_{i_n}, a))$. By Condition 2, the latter is equal to $\alpha(B_{i_1}, a) \cup \cdots \cup \alpha(B_{i_n}, a) = \alpha(B_{i_1} \cup \cdots \cup B_{i_n}, a) = \alpha(S(\beta(B_i, w), a))$. By the inductive assumption, we get $\alpha(S(\beta(B_i, w), a)) = \alpha(a(B_i, w), a) = \alpha(B_i, wa)$, which proves our claim.

**Proof.** (Theorem 6) First we prove that any NFA $\mathcal{B}$ satisfying Conditions 1–3 is an atomic NFA of $L$. Let $B_i \in B$ be a state of $\mathcal{B}$. If $w \in L_{B_i, B_F}(\mathcal{B})$, then by Condition 3, there exists $B_j \in B_i$ such that $A_{p-1} \in B_j$, and we have $A_{p-1} \in S(\beta(B_i, w))$. By Lemma 7, we get $A_{p-1} \in \alpha(B_i, w)$, implying that there is some $A_k \in B_i$ such that $w \in L_{A_k, (A_{p-1})}(\mathcal{B})$. Conversely, if $w \in L_{A_k, (A_{p-1})}(\mathcal{B})$ and $A_k \in B_i$, then $A_{p-1} \in \alpha(B_i, w) = S(\beta(B_i, w))$. Hence there exists $B_j \in \beta(B_i, w)$ such that $A_{p-1} \in B_j$. Consequently, every word accepted in $\mathcal{B}$ from state $B_i$ is in some atom $A_k$ such that $A_k \in B_i$, and every word in an atom $A_k$ such that $A_k \in B_i$ is also in $L_{B, B_F}(\mathcal{B})$. Therefore the right language of $B_i$ in $\mathcal{B}$ is equal
to the union of atoms $A_k$ such that $A_k \in B_i$. In particular, $L_{B_i,B_F}(\mathcal{B})$ is the union of atoms whose atom symbols appear in the initial collection of $\mathcal{B}$ which, by Condition 1, is the same as the union of atoms whose atom symbols are initial in $\mathcal{A}$. But that last union is precisely $L_{A_1,\{A_{p-1}\}}(\mathcal{A}) = L$. Since any two sets $B_i$ and $B_j$ are different, and atoms are disjoint, $\mathcal{B}$ is reduced. Hence $\mathcal{B}$ is a reduced atomic NFA of $L$.

Conversely, we show that if $\mathcal{B}$ is a reduced atomic NFA of $L$, then it must satisfy Conditions 1–3. We assume that $\mathcal{B}$ is atomic, that is, for every state $B_i$ of $\mathcal{B}$, the right language of $B_i$ is equal to the union of atoms $A_k$ such that $A_k \in B_i$.

For Condition 1, let $A_j \in S(B_j)$. Then there is a state $B_k \in B_j$ such that $A_j \in B_k$. So for any $w \in A_j$, $w \in L(\mathcal{B})$. Since $L(\mathcal{B}) = L(\mathcal{A})$, we have $w \in L(\mathcal{A})$ for all $w \in A_j$. Thus $A_j \in A_1$. Conversely, if $A_j \in A_1$, then for all $w \in A_j$, $w \in L(\mathcal{A}) = L(\mathcal{B})$. Since $\mathcal{B}$ is atomic, there is an initial state $B_k$ such that $A_j \subseteq L_{B_k,B_F}(\mathcal{B})$. Hence $A_j \in S(B_k)$.

For Condition 2, if $A_j \in S(\beta(B_i,a))$, then $L_{B_i,B_F}(\mathcal{B})$ must contain $aA_j$. So there exists some $A_k \in B_i$ such that $aA_j \subseteq A_k$. Thus $A_j \in \alpha(B_i,a)$. Conversely, if $A_j \in \alpha(B_i,a)$, then there is an atom $A_k \in B_i$ such that $A_j \subseteq \alpha(A_k,a)$, implying $aA_j \subseteq A_k$. Since $A_k \in B_i$, $L_{B_i,B_F}(\mathcal{B})$ must contain $aA_j$. Hence $A_j \in S(\beta(B_i,a))$.

For Condition 3, we first suppose that $B_i \in \mathcal{B}_F$. Then $\varepsilon$ is in the right language of $B_i$. Since $\mathcal{B}$ is atomic, $\varepsilon$ must be in one of the atoms of $B_i$. However, the only atom containing $\varepsilon$ is $A_{p-1}$, so $A_{p-1} \in B_i$. Conversely, if $A_{p-1} \in B_i$, then $\varepsilon$ is in the right language of $B_i$, and $B_i$ is a final state by definition of an NFA.

The number of trim reduced atomic NFAs can be very large. There can be such NFAs with as many as $2^p - 1$ non-empty states, since there are that many non-empty sets of positive atoms. In the general case, however, not all sets of positive atom symbols can be states of an atomic NFA. The largest reduced atomic NFA is characterized in the following theorem.

**Theorem 8 (Maximal Atomic NFA)** If $\mathcal{B}$ is the collection of all sets $B_i$ such that $B_i$ is a non-empty subset of the set of positive atom symbols $\{A_k \mid A_k \subseteq K_j\}$ of some quotient $K_j$ of $L$, then there exists a trim reduced atomic NFA of $L$ with state set $\mathcal{B}$.

**Proof.** Let $\mathcal{B} = (B, \Sigma, \beta, B_I, B_F)$ be an NFA in which the state set $\mathcal{B}$ is the collection of all sets $B_i$ such that $B_i$ is a non-empty subset of the set of atom symbols $\{A_k \mid A_k \subseteq K_j\}$ of some quotient $K_j$ of $L$, where $j \in \{0, \ldots, n - 1\}$, $\beta(B_i, a) = \{B_j \mid B_j \subseteq \alpha(B_i,a)\}$ for every $B_i \in B$ and $a \in \Sigma$, $B_i \in B_I$ if and only if $B_i$ is a subset of the set of atom symbols of the initial quotient $K_{in}$, and $B_i \in B_F$ if and only if $A_{p-1} \in B_i$. We claim that $\mathcal{B}$ is a trim reduced atomic NFA of $L$.

We show that $\mathcal{B}$ is trim. Consider any state $B_i$ of $\mathcal{B}$. Let $K_j$ be a quotient such that $B_i$ is a subset of the set of atom symbols of $K_j$, and let $B_j$ be the set of atom symbols corresponding to $K_j$. Let $B_0$ be the set of atom symbols corresponding to the initial quotient $K_{in}$ of $L$. Note that $B_0 = A_I$. Since every set of atom symbols
corresponding to some quotient is reachable from the initial set of atom symbols in the atomaton \( \mathfrak{A} \), there must be a word \( w \in \Sigma^* \), such that \( B_j \) is reachable from \( B_0 \) by \( w \) in \( \mathfrak{A} \). We show that \( B_i \) is reachable from some initial state of \( \mathfrak{B} \) by \( w \). If \( w = \varepsilon \), then \( K_j = K_{in} \), and since \( B_i \subseteq B_j \), it follows that \( B_i \) is an initial state of \( \mathfrak{B} \) reachable from itself by \( \varepsilon \). If \( w = au \) for some \( u \in \Sigma^* \) and \( a \in \Sigma \), then there is a state \( B_0 \) of \( \mathfrak{B} \), reachable from \( B_0 \) by \( u \), such that \( B_u \) corresponds to the quotient \( u^{-1}L \) of \( L \) and \( B_j = \alpha(B_u, a) \). Since \( B_i \subseteq B_j \) and \( B_j = \alpha(B_u, a) \), by the definition of \( \beta \) we have \( B_i \in \beta(B_u, a) \). Thus, \( B_i \) is reachable from \( B_0 \) in \( \mathfrak{B} \) by \( au \).

We also have to show that there is a word \( w \in \Sigma^* \), such that some final state of \( \mathfrak{B} \) is reachable from \( B_i \) by \( w \). If \( B_i \) is final, then it is reachable from itself by \( w = \varepsilon \). If \( B_i \) is not final, then consider any \( A_k \in B_i \). Since the right language of the state \( A_k \) in the atomaton \( \mathfrak{A} \) is not empty, and \( A_k \) cannot be the final state of \( \mathfrak{A} \), there must be some state \( A_l \) of \( \mathfrak{A} \) and some \( a \in \Sigma \), such that \( A_l \in \alpha(A_k, a) \). Now we know that there is some \( B_j \) such that \( A_l \in B_j \) and \( \alpha(B_l, a) = B_j \). Since \( \beta(B_l, a) \) is the collection of all non-empty subsets of \( B_j \), it follows that \( \{A_l\} \in \beta(B_l, a) \).

Since the final state \( A_{p-1} \) of \( \mathfrak{A} \) is reachable from \( A_1 \) by any word \( v \in A_1 \), we get \( \{A_{p-1}\} \in \beta(B_1, av) \) by the definition of \( \beta \). So a final state \( \{A_{p-1}\} \) of \( \mathfrak{B} \) is reachable from \( B_i \) by \( av \). Thus, \( \mathfrak{B} \) is trim.

To see that \( \mathfrak{B} \) is a reduced atomic NFA, one verifies that Conditions 1–3 of Theorem 6 hold. Thus by Theorem 6, \( \mathfrak{B} \) is a trim reduced atomic NFA of \( L \).}

**Theorem 9 (NFA with \( 2^p - 1 \) States)** A regular language \( L \) has a trim reduced atomic NFA with \( 2^p - 1 \) states if and only if there is some quotient \( K_i \) of \( L \), such that \( K_i = A_0 \cup \cdots \cup A_{p-1} \).

**Proof.** Let \( \mathfrak{B} = (B, \Sigma, \beta, B_0, B_F) \) be a trim reduced atomic NFA of \( L \) with \( 2^p - 1 \) states. Then there must be a state \( B_i \) of \( \mathfrak{B} \) such that \( B_i = \{A_0, \ldots, A_{p-1}\} \). Since the right language of any state of a trim NFA is a subset of some quotient, we have \( L_{B_i, B_F}(\mathfrak{B}) = A_0 \cup \cdots \cup A_{p-1} \subseteq K_i \) for some quotient \( K_i \) of \( L \). On the other hand, \( K_i \) must be a union of some atoms, so we get \( K_i = A_0 \cup \cdots \cup A_{p-1} \).

Conversely, let \( K_i = A_0 \cup \cdots \cup A_{p-1} \) be a quotient of \( L \) which includes all the positive atoms of \( L \). Then by Theorem 8, there is a trim reduced atomic NFA of \( L \) in which the state set is the collection of all non-empty subsets of the set of positive atom symbols. This NFA has \( 2^p - 1 \) states.

A minimal atomic NFA of a language \( L \) can possibly be as small as a minimal NFA for \( L \). An example of a language with this property is any language where a minimal DFA is also a minimal NFA as is the case, for instance, of the language \( L_2 \) with the minimal DFA \( D_2 \) of Example 3.

4. Atom Complexity

The quotient complexity [2] of \( L \) is the number of quotients of \( L \), and this is the same number as the number of states in the minimal DFA recognizing \( L \); the latter
number is known as the state complexity [10] of \( L \). Quotient complexity allows us to use language-theoretic methods, whereas state complexity is more amenable to automaton-theoretic techniques. We use one of these two points of view or the other, depending on convenience.

It has been suggested by Brzozowski and Ye [6] that syntactic complexity can be a useful measure of complexity. It has its roots in the syntactic congruence \( \approx_L \) defined by a language \( L \subseteq \Sigma^* \) as follows: For \( x, y \in \Sigma^* \),

\[ x \approx_L y \text{ if and only if } uxv \in L \iff uyv \in L \text{ for all } u, v \in \Sigma^* . \]

The syntactic semigroup of \( L \) is the quotient semigroup \( \Sigma^+ / \approx_L \). Syntactic complexity is the cardinality of the syntactic semigroup. This complexity may be able to distinguish two regular languages with the same quotient complexity. For example [6], a language with three quotients may have syntactic complexity as low as 3 or as high as 27. The syntactic semigroup is isomorphic to the semigroup of transformations of the set of states, called the transition semigroup, by non-empty words in the minimal DFA of \( L \). The transition semigroup is often used to represent the syntactic semigroup.

Our main result concerns the quotient complexity of atoms of regular languages, which represents yet another complexity measure. We say that a language has maximal atom complexity if (a) it has all \( 2^n \) atoms, and (b) they all reach their maximal bounds, as stated below.

For \( n = 1 \), there is only one non-empty language, \( \Sigma^* \); it has one atom, \( \Sigma^* \), which has quotient complexity 1. From now on we consider only \( n \geq 2 \).

**Theorem 10 (Atom Complexity)** Let \( L \subseteq \Sigma^* \) be a non-empty regular language and let its set of quotients be \( \mathcal{K} = \{K_0, \ldots, K_{n-1}\} \). The quotient complexity of the atoms with 0 or \( n \) complemented quotients is less than or equal to \( 2^n - 1 \). For \( r \) satisfying \( 1 \leq r \leq n - 1 \), the quotient complexity of any atom of \( L \) with \( r \) complemented quotients is less than or equal to

\[
f(n, r) = 1 + \sum_{k=1}^{r} \sum_{h=k+1}^{n} \binom{n}{h} \frac{n^h}{k^h} .
\]

The atoms of the language \( L_n \) of the DFA \( D_n \) of Fig. 2 meet the bounds given above.

The proof of this result is postponed to Secs. 5–8.

![Fig. 2. DFA \( D_n \) of language \( L_n \) whose atoms meet the bounds.](image-url)
The following relations exist among the three complexity measures [3]:

**Theorem 11 (Syntactic Semigroup, Atoms and Reversal)**

Maximal syntactic complexity of a regular language implies maximal atom complexity, but the converse is false. Also, maximal atom complexity implies maximal complexity \(2^n\) of reversal, but the converse is false.

Thus atom complexity defines a new complexity class of regular languages. These results provide additional motivation for studying the complexity of atoms.

5. **Upper Bounds on the Quotient Complexities of Atoms**

We now derive upper bounds on the quotient complexity of atoms. First we deal with the two atoms that have only uncomplemented or only complemented quotients.

Let \(L \subseteq \Sigma^*\) be a non-empty regular language and let its set of quotients be \(K = \{K_0, \ldots, K_{n-1}\}\), with \(n \geq 2\).

**Proposition 12 (Atoms with 0 or \(n\) Complemented Quotients)** The quotient complexity of the two atoms \(A_K = K_0 \cap \cdots \cap K_{n-1}\) and \(A_\emptyset = K_0 \cap \cdots \cap K_{n-1}\) is less than or equal to \(2^n - 1\).

**Proof.** Each quotient \(w^{-1}A_K\) is the intersection of languages \(w^{-1}K_i\), which are quotients of \(L\): \(w^{-1}A_K = w^{-1}(K_0 \cap \cdots \cap K_{n-1}) = w^{-1}K_0 \cap \cdots \cap w^{-1}K_{n-1}\). Since these quotients of \(L\) need not be distinct, \(w^{-1}A_K\) may be the intersection of any non-empty subset of quotients of \(L\). Hence \(A_K\) can have at most \(2^n - 1\) quotients.

The argument for the atom \(A_\emptyset = K_0 \cap \cdots \cap K_{n-1}\) with \(n\) complemented quotients is similar, since \(w^{-1}K_i = w^{-1}K_i\).

Next, we present an upper bound on the quotient complexity of any atom with at least one and fewer than \(n\) complemented quotients.

**Proposition 13 (Atoms with \(r\) Complemented Quotients, \(1 \leq r \leq n - 1\))**

For \(1 \leq r \leq n - 1\), the quotient complexity of any atom with \(r\) complemented quotients is less than or equal to

\[
f(n, r) = 1 + \sum_{k=1}^{r} \sum_{h=k+1}^{k+n-r} \binom{n}{h} \binom{h}{k}.
\]

**Proof.** Consider an intersection of complemented and uncomplemented quotients that constitutes an atom. Without loss of generality, we arrange the terms in the intersection in such a way that all complemented quotients appear on the right. Thus let \(A_i = K_0 \cap \cdots \cap K_{n-r-1} \cap (K_{n-r} \cap \cdots \cap K_{n-1})\) be an atom of \(L\) with \(r\) complemented quotients, where \(1 \leq r \leq n - 1\). The quotient of \(A_i\) by any \(w \in \Sigma^*\) is

\[
w^{-1}A_i = w^{-1}(K_0 \cap \cdots \cap K_{n-r-1} \cap (K_{n-r} \cap \cdots \cap K_{n-1}))
= w^{-1}K_0 \cap \cdots \cap w^{-1}K_{n-r-1} \cap w^{-1}K_{n-r} \cap \cdots \cap w^{-1}K_{n-1}.
\]
Since each quotient \( w^{-1}K_j \) is a quotient, say \( K_i,j \), of \( L \), we have

\[
w^{-1}A_i = K_{i_n} \cap \cdots \cap K_{i_{n-r-1}} \cap K_{i_{n-r}} \cap \cdots \cap K_{i_{n-1}}.
\]

The cardinality of a set \( S \) is denoted by \( |S| \). Let the set of distinct quotients of \( L \) appearing in \( w^{-1}A_i \) uncomplemented (respectively, complemented) be \( X \) (respectively, \( Y \)), where \( 1 \leq |X| \leq n - r \) and \( 1 \leq |Y| \leq r \). If \( X \cap Y \neq \emptyset \), then \( w^{-1}A_i = \emptyset \). Consider now the case where \( X \cap Y = \emptyset \), and let \( |X \cup Y| = h \), where \( 2 \leq h \leq n \); there are \( \binom{n}{h} \) such sets \( X \cup Y \). Suppose further that \( |Y| = k \), where \( 1 \leq k \leq r \). There are \( \binom{k}{h} \) ways of choosing \( Y \). Hence there are at most \( \sum_{h=k+1}^{k+n-r} \binom{n}{h} \binom{k}{h} \) distinct intersections with \( k \) complemented quotients. Thus, the total number of intersections of uncomplemented and complemented quotients can be at most \( \sum_{k=1}^{k+n-r} \binom{n}{k} \binom{k}{h} \). Adding 1 for the empty quotient of \( w^{-1}A_i \), we get the required bound.

We now consider the properties of the function \( f(n, r) \).

**Proposition 14 (Properties of Bounds)** For \( 1 \leq r \leq n-1 \), the function \( f(n, r) \) of Eq. (1) satisfies the following properties:

1. \( f(n, r) = f(n, n-r) \).
2. For a fixed \( n \), the maximal value of \( f(n, r) \) occurs when \( r = \lfloor n/2 \rfloor \).

**Proof.** Since \( f(n, r) = 1 + \sum_{k=1}^{r} \binom{n}{h} \binom{k}{h} \binom{h}{k} \binom{k}{l} \binom{l}{m} \binom{m}{l} \), and the following equations hold:

\[
\sum_{l=1}^{r} \sum_{h=1}^{k} \binom{n}{h} \binom{k}{h} \binom{h}{k} = \sum_{l=1}^{r} \sum_{h=1}^{k} \binom{n}{k+l} \binom{k}{k} = \sum_{l=1}^{r} \sum_{h=1}^{k} \binom{n}{k+l} \binom{k}{k} = \sum_{l=1}^{r} \sum_{h=1}^{k} \binom{n}{k+l} \binom{k}{k} = \sum_{l=1}^{r} \sum_{h=1}^{k} \binom{n}{k+l} \binom{k}{k}
\]

we have \( f(n, r) = f(n, n-r) \).

For the second part, we assume that \( 1 \leq r < \lfloor n/2 \rfloor \) holds. We will show that \( f(n, r+1) > f(n, r) \) for this case. After some straightforward rewriting we find that \( f(n, r+1) - f(n, r) \) is equal to

\[
\sum_{h=r+2}^{r+n} \binom{n}{h} \binom{h}{r+1} + \sum_{k=1}^{r} \binom{n}{k+n-r} \binom{k+n-r}{k} \binom{k+n-r}{k} - \sum_{k=1}^{r} \binom{n}{k+n-r} \binom{k+n-r}{k}.
\]

Assuming \( 1 \leq k \leq l \), we will show that \( \binom{k+n-r}{k+r+1} > \binom{k+n-r}{k} \). We consider

\[
\frac{\binom{k+n-r}{k+r+1}}{\binom{k+n-r}{k}} = \frac{(k+n-r)!}{(r+1)!(k+n-2r-1)!} \cdot \frac{(k+n-r)!}{k!(n-r)!} = \frac{k!(n-r)!}{(r+1)!(k+1)!(n-2r+k-1)!} = \frac{\binom{n-r}{r+1} \cdots \binom{n-2r+k}{k+1}}{r+1} \cdot \frac{\binom{n-2r+k}{k}}{k+1}.
\]
The condition $1 \leq r < [n/2]$ implies that $n > 2r + 1$; consequently we have

$$n - r > r + 1, \quad n - r - 1 > r, \ldots, \quad n - 2r + k > k + 1.$$ 

Therefore $(k^{r+1}n)/k^{r+1} > 1$, which implies that $(k^{r+1}n)/k > (k^{r+1}n)/k$. 

It follows that $\sum_{k=1}^r (k^{r-1}n/k^{r+1}) > \sum_{k=1}^r (k^{r-1}n/k^{r+1})$, and $f(n, r + 1) - f(n, r) > 0$. So, if $1 \leq r < [n/2]$, then $f(n, r + 1) > f(n, r)$. Since $f(n, r) = f(n, n - r)$, the maximum of $f(n, r)$ occurs when $r = [n/2]$. 

Some numerical values of $f(n, r)$ are shown in Table 1. The figures in boldface type are the maxima for a fixed $n$. The row marked max shows the maximal quotient complexity of the atoms of $L$. The row marked ratio shows the value of $f(n, [n/2])/f(n, 1)$, for $n \geq 2$.

The following observations are from Volker Diekert (personal communication). For $r = [n/2]$ the difference $3^n - f(n, r)$ grows as $8^{n/2}$. Hence the ratio $f(n, r)/f(n - 1, r)$ converges to 3. The ratio oscillates around 3: a combinatorial interpretation shows that for $n \geq 10$, we have $f(n, r)/f(n - 1, r) > 3$ if $n$ is even, and $f(n, r)/f(n - 1, r) < 3$ if $n$ is odd.

<table>
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<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
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<th>10</th>
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<td>*</td>
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<td>141</td>
<td>406</td>
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<tr>
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<td>*</td>
<td>7</td>
<td>29</td>
<td>141</td>
<td>504</td>
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<td>*</td>
<td>*</td>
<td>*</td>
<td>15</td>
<td>76</td>
<td>406</td>
<td>1,548</td>
<td>5,083</td>
<td>15,361</td>
<td>44,071</td>
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<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>31</td>
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<td>1,086</td>
<td>4,425</td>
<td>15,361</td>
<td>48,733</td>
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<tr>
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<td>48,733</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>ratio</td>
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<td>3.02</td>
<td>3.17</td>
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</table>

6. Another Representation of Atomata

The next theorem, a slightly modified version of a result from [1] also discussed in [4], will be used several times.

**Theorem 15 (Determinization)** If an NFA $\mathfrak{A}$ has no empty states and $\mathfrak{R}^R$ is deterministic, then $\mathfrak{R}^D$ is minimal.

We prove\(^\ast\) that $\mathfrak{A}$ is isomorphic to $\mathfrak{D}^{RDR}$. We deal with the following automata:

1. Quotient DFA $\mathfrak{D} = (\mathcal{K}, \Sigma, \delta, \mathcal{K}_m, F)$ of $L$ whose states are quotient symbols.
2. The reverse $\mathcal{D}^R = (\mathcal{K}, \Sigma, \delta^R, F, \{\mathcal{K}_m\})$ of $\mathfrak{D}$. The states in $\mathcal{K}$ are still quotient symbols, but their right languages are no longer quotients of $L$.

\(^\ast\) It was shown in [4] that the atomaton $\mathfrak{A}$ of $L$ with reachable atoms only is isomorphic to the trim version of $\mathfrak{D}^{RDR}$, where $\mathfrak{D}$ is the quotient DFA of $L$. 

(3) The determinized reverse $D^{RD} = (S, \Sigma, \gamma, F, G)$, where $S \subseteq 2^K$ and $G = \{S_i \in S \mid K_{i0} \cap K_{i1} \cap \cdots \cap K_{in} \cap \cdots \cap K_{im} \in S_i\}$. The states in $S$ are sets of quotient symbols, i.e., subsets of $K$. Since $(D^R)^R = D$ is deterministic and all of its states are reachable, $D^R$ has no empty states. By Theorem 15, DFA $D^{RD}$ is minimal and accepts $L^R$; hence it is isomorphic to the quotient DFA of $L^R$.

(4) The reverse $D^{RDR} = (S, \Sigma, \gamma^R, G, \{F\})$ of $D^{RD}$; here the states are still sets of quotient symbols.

(5) The atomaton $A = (A, \Sigma, \alpha, A_1, \{A_p\})$, whose states are atom symbols.

(6) The reverse $A^R = (A, \Sigma, \alpha^R, A_{p-1}, A_1)$ of $A$, whose states are still atom symbols, though their right languages are no longer atoms.

The results from [4] and our new definition of atoms imply that $A^R$ is a minimal DFA that accepts $L^R$. It follows that $A^R$ is isomorphic to $D^{RD}$. Our next result makes this isomorphism precise.

**Proposition 16 (Isomorphism)** Let $\varphi : A \to S$ be the mapping assigning to state $A_j$, given by $A_j = K_{i0} \cap \cdots \cap K_{in} \cap \cdots \cap K_{im}$ of $\mathfrak{A}$, the set $\{K_{i0}, \ldots, K_{in-1}\}$. Then $\varphi$ is a DFA isomorphism between $A^R$ and $D^{RD}$.

**Proof.** The initial state $A_{p-1}$ of $A$ is mapped to the set of all quotients containing $\varepsilon$, which is precisely the initial state $F$ of $D^{RD}$. Since the quotient $L$ appears unimplemented in every initial atom $A_i \in I$, the image $\varphi(A_i)$ contains $L$. Thus the set of final states of $A^R$ is mapped to the set of final states of $D^{RD}$.

It remains to be shown that for all $A_i, A_j \in A$ and $a \in \Sigma$, we have $\alpha^R(A_j, a) = A_i$ if and only if $\gamma(\varphi(A_j), a) = \varphi(A_i)$.

Consider atom $A_i$ with $P_i$ as the set of quotients that appear unimplemented in $A_i$. Also define the corresponding set $P_j$ for $A_j$. If there is a missing quotient $K_k$ in the intersection $a^{-1}A_i$, we use $a^{-1}A_i \cap (K_k \cup K_{k'})$. We do this for all missing quotients until we obtain a union of atoms. Hence $A_j \in \alpha(A_i, a)$ can hold in $A$ if and only if $P_j \supseteq \delta(P_i, a)$ and $P_j \cap \delta(Q \setminus P_i, a) = \emptyset$. It follows that in $A^R$ we have $\alpha^R(A_j, a) = A_i$ if and only if $P_j \supseteq \delta(P_i, a)$ and $P_j \cap \delta(Q \setminus P_i, a) = \emptyset$.

Now consider $D^{RD}$. Let $P_j$ be any subset of $Q$; then the successor set of $P_i$ in $D$ is $\delta(P_i, a)$. Let $\delta(P_i, a) = P_k$. So in $D^R$, we have $P_k \supseteq \delta^R(P_i, a)$. But suppose that state $q$ is not in $\delta(Q, a)$; then $\delta^R(q, a) = \emptyset$. Thus we also have $P_i \supseteq \delta^R(P_k \cup \{q\}, a)$. So for any $P_j$ containing $\delta(P_i, a)$ and satisfying $P_j \cap \delta(Q \setminus P_i, a) = \emptyset$, $\gamma(P_j, a) = P_i$.

We have now shown that $\alpha^R(A_j, a) = A_i$ if and only if $\gamma(\varphi(A_j), a) = \varphi(A_i)$. $\Box$

**Corollary 17.** The mapping $\varphi$ is an NFA isomorphism between $\mathfrak{A}$ and $D^{RDR}$. 

In the remainder of the paper it is more convenient to use the $D^{RDR}$ representation of atomata, rather than that of Definition 2.
7. The Witness Languages and Automata

We now formally introduce a class \( \{ L_n \mid n \geq 2 \} \) of regular languages defined by the quotient DFAs \( \mathcal{D}_n \) given below; we shall prove that the atoms of each language \( L_n = L(\mathcal{D}_n) \) in this class meet the worst-case quotient complexity bounds.

**Definition 18 (Witness)** For \( n \geq 2 \), let \( \mathcal{D}_n = (Q, \Sigma, \delta, q_0, F) \), where \( Q = \{ 0, \ldots, n-1 \} \), \( \Sigma = \{ a, b, c \} \), \( \delta(i, a) = i + 1 \mod n \), \( \delta(0, b) = 1 \), \( \delta(1, b) = 0 \), \( \delta(i, b) = i \) for \( i > 1 \), \( \delta(i, c) = i \) for \( 0 \leq i < n-2 \), and \( \delta(n-1, c) = 0 \), \( q_0 = 0 \), and \( F = \{ n-1 \} \). Let \( L_n \) be the language accepted by \( \mathcal{D}_n \).

For \( n \geq 3 \), the DFA \( \mathcal{D}_n \) of Definition 18 is illustrated in Fig. 2, and \( \mathcal{D}_2 \) is the DFA of Example 3 (\( a \) and \( b \) coincide). DFA \( \mathcal{D}_n \) is minimal, since for \( 0 \leq i \leq n-1 \), state \( i \) accepts \( a^{n-1-i} \), and no other state accepts this word.

A transformation of a set \( Q \) is a mapping of \( Q \) into itself. The set of all transformations of a finite set \( Q \) is a semigroup under composition, in fact, a monoid \( T_Q \) of \( n^3 \) elements. A permutation of \( Q \) is a mapping of \( Q \) onto itself. A transposition, \( (i, j) \), interchanges \( i \) and \( j \) but does not affect any other element. A unitary transformation, \( (i \to j) \), changes \( i \) to \( j \) but does not affect any other element.

The following is well known:

**Theorem 19 (Transformations)** The transformation monoid \( T_Q \) can be generated by any cyclic permutation of \( n \) elements together with any transposition and any unitary transformation.

In any DFA \( \mathcal{D} = (Q, \Sigma, \delta, q_0, F) \), each word \( w \) in \( \Sigma^+ \) performs a transformation on \( Q \) defined by \( \delta(\cdot, w) \). The set of all these transformations is the transition semigroup of \( \mathcal{D} \). By Theorem 19, the transition semigroup of our witness \( \mathcal{D}_n \) has \( n^n \) elements, since \( a \) is a cyclic permutation, \( b \) is a transposition and \( c \) is a unitary transformation.

The following is a result of Salomaa, Wood and Yu [8] concerning reversal:

**Theorem 20 (Transformations and Reversal)** Let \( \mathcal{D} \) be a minimal DFA with \( n \geq 2 \) states accepting a language \( L \). If the transition semigroup of \( \mathcal{D} \) has \( n^n \) elements, then the quotient complexity of \( L^R \) is \( 2^n \).

**Corollary 21 (Reversal)** For \( n \geq 2 \), the quotient complexity of \( L_n^R \) is \( 2^n \).

**Corollary 22 (Number of Atoms of \( L_n \))** The language \( L_n \) has \( 2^n \) atoms.

**Proof.** By Corollary 17, the atomaton of \( L_n \) is isomorphic to the reversed quotient DFA of \( L_n^R \). By Corollary 21, the quotient DFA of \( L_n^R \) has \( 2^n \) states, and so the empty set of states of \( L_n \) is reachable in \( L_n^R \). Hence \( L_n^R \) has the empty quotient, implying that the intersection of all the complemented quotients of \( L_n \) is non-empty, and so \( L_n \) has \( 2^n \) atoms.
Proposition 23 (Transitions of the Atomaton) Let $\mathcal{D}_n = (Q, \Sigma, \delta, q_0, F)$ be the DFA of Definition 18. The atomaton of $L_n = L(\mathcal{D}_n)$ is the NFA $\mathfrak{A}_n = (2^{Q}, \Sigma, \alpha, I, \{n - 1\})$, where

1. If $S = \{\emptyset\}$, then $\alpha(S, a) = \{\emptyset\}$. Otherwise, 
   $\alpha(\{s_1, \ldots, s_k\}, a) = \{s_1 + 1, \ldots, s_k + 1\}$, where the addition is modulo $n$.

2. If $\{0, 1\} \cap S = \emptyset$, then 
   \begin{align*} 
   (a) & \quad \alpha(S, b) = S, \\
   (b) & \quad \alpha(\{0\} \cup S, b) = \{1\} \cup S, \\
   (c) & \quad \alpha(\{1\} \cup S, b) = \{0\} \cup S, \\
   (d) & \quad \alpha(\{0, 1\} \cup S, b) = \{0, 1\} \cup S. 
   \end{align*}

3. If $\{0, n - 1\} \cap S = \emptyset$, then 
   \begin{align*} 
   (a) & \quad \alpha(S, c) = \{S, \{n - 1\} \cup S\}, \\
   (b) & \quad \alpha(\{0, n - 1\} \cup S, c) = \{\{0, n - 1\} \cup S, \{0\} \cup S\}, \\
   (c) & \quad \alpha(\{0\} \cup S, c) = \emptyset, \\
   (d) & \quad \alpha(\{n - 1\} \cup S, c) = \emptyset. 
   \end{align*}

Proof. The reverse of DFA $\mathcal{D}_n$ is the NFA $\mathcal{D}_n^R = (Q, \Sigma, \delta^R, \{n - 1\}, \{0\})$, where $\delta^R$ is defined by $\delta^R(i, a) = i - 1 \mod n$, $\delta^R(i, b) = \delta(i, b)$, $\delta^R(0, c) = \{0, n - 1\}$, $\delta^R(n - 1, c) = \emptyset$, and $\delta^R(i, c) = i$, for $0 < i < n - 1$. After applying determinization and reversal to $\mathcal{D}_n^R$, the claims follow by Corollary 17.

8. Tightness of the Upper Bounds

We now show that the upper bounds derived in Sec. 5 are tight by proving that the atoms of the language $L_n$ of Definition 18 meet those bounds.

Since the states of any atomaton $\mathfrak{A}_n = (A, \Sigma, \alpha, A_I, \{A_{p-1}\})$ are atom symbols $A_i$, and the right language of each $A_i$ is an atom $A_i$, the languages $A_i$ are properly represented by the atomaton. Since, however, the atomaton is an NFA, to find the quotient complexity of $A_i$, we need the equivalent minimal DFA.

Let $\mathcal{D}_n$ be the $n$-state quotient DFA of Definition 18 for $n \geq 2$, and recall that $L(\mathcal{D}_n) = L_n$. In the sequel, using Corollary 17, we represent the atomaton $\mathfrak{A}_n$ of $L_n$ by the isomorphic NFA $\mathcal{D}_n^{RDR} = (S, \Sigma, \gamma^R, G, \{F\})$, and identify the atoms by their sets of uncomplemented quotients. We represent atoms by the subsets of the quotients, that is, by subsets of $Q = \{0, \ldots, n - 1\}$, as in Definition 18.

In this framework, to find the quotient complexity of an atom $A_P$, with $P \subseteq Q$, we start with the NFA $\mathfrak{A}_P = (S, \Sigma, \gamma^R, \{P\}, \{F\})$, which has the same states, transitions, and final state as the atomaton, but has only one initial state $P$ corresponding to the atom symbol $A_P$. Because $\mathfrak{A}_P^D$ is deterministic and $\mathfrak{A}_P$ has no empty states, $\mathfrak{A}_P^D$ is minimal by Theorem 15. Therefore, $\mathfrak{A}_P^D$ is the quotient DFA of the atom $A_P$. The states of $\mathfrak{A}_P^D$ are certain sets of sets of quotient symbols; to reduce confusion we refer to them as collections of sets.
The particular collections appearing in $\mathfrak{A}_Q^D$ will be called “intervals”. Let $U$ be a subset of $Q$ with $|U| = u$, and let $V$ be a subset of $U$ with $|V| = v$. Define $[V, U]$ to be the collection of all $2^{u-v}$ subsets of $U$ containing $V$. There are $\binom{u}{v}$ collections of the form $[V, U]$, because there are $\binom{u}{v}$ ways of choosing $U$, and for each such choice there are $\binom{v}{w}$ ways of choosing $V$. The collection $[V, U]$ is called the interval between $V$ and $U$. The type of an interval $[V, U]$ is the ordered pair $(v, u)$.

The following result is well-known:

**Theorem 24 (Permutations)** The symmetric group of size $n!$ of all permutations of a set $Q = \{0, \ldots, n-1\}$ is generated by any cyclic permutation of $Q$ together with any transposition.

**Lemma 25 (Strong-Connectedness of Intervals)** Intervals of the same type are strongly connected by words in $(a, b)^*$. 

**Proof.** Let $[V_1, U_1]$ and $[V_2, U_2]$ be any two intervals of the same type. Arrange the elements of $V_1$ in increasing order, and do the same for the elements of the sets $V_2$, $U_1 \setminus V_1$, $U_2 \setminus V_2$, $Q \setminus U_1$, and $Q \setminus U_2$. Let $\pi: Q \to Q$ be the mapping that assigns the $i$th element of $V_2$ to the $i$th element of $V_1$, the $i$th element of $U_2 \setminus V_2$ to the $i$th element of $U_1 \setminus V_1$, and the $i$th element of $Q \setminus U_2$ to the $i$th element of $Q \setminus U_1$. For any $R_1$ such that $V_2 \subseteq R_1 \subseteq U_1$, there is a corresponding subset $R_2 = \pi(R_1)$, where $V_2 \subseteq R_2 \subseteq U_2$. Thus $\pi$ establishes a one-to-one correspondence between the elements of the intervals $[V_1, U_1]$ and $[V_2, U_2]$. Also, $\pi$ is a permutation of $Q$ and can be performed by a word $w \in (a, b)^*$ in $\mathfrak{D}_n$, by Theorem 24. Thus every set $R_2$ above is reachable from $R_1$ by $w$. So $[V_2, U_2]$ is reachable from $[V_1, U_1]$.

**Lemma 26 (Reachability)** Let $[V, U]$ be any interval of type $(v, u)$. If $v \geq 2$, then from $[V, U]$ we can reach an interval of type $(v-1, u)$. If $u \leq n-2$, then from $[V, U]$ we can reach an interval of type $(v, u+1)$.

**Proof.** If $v \geq 2$, then by Lemma 25, from $[V, U]$ we can reach an interval $[V', U']$ of type $(v, u)$ such that $\{0, n-1\} \subseteq V'$. By input $c$ we reach $[V' \setminus \{n-1\}, U']$ of type $(v-1, u)$. For the second claim, if $u \leq n-2$, then by Lemma 25, from $[V, U]$ we can reach an interval $[V', U']$ of type $(v, u)$ such that $\{0, n-1\} \cap V' = \emptyset$. By input $c$ we reach $[V', U' \cup \{n-1\}]$ of type $(v, u+1)$.

**Proposition 27 (Atoms with $0$ or $n$ Complemented Quotients)** The quotient complexity of the atoms $A_Q$ and $A_0$ of $L_n$ is $2^n - 1$.

**Proof.** Let $\mathfrak{A}_Q^D$ be the modified atomaton with only one initial state, $Q (\emptyset)$. By the arguments above, $\mathfrak{A}_Q^D (\mathfrak{A}_0^D)$ is the quotient DFA of $A_Q (A_0)$; hence it suffices to prove the reachability of $2^n - 1$ collections.

For $A_Q$, the initial state of $\mathfrak{A}_Q^D$ is the collection $\{Q\}$, which is the interval $[Q, Q]$. Now suppose that we have reached an interval of type $(v, u)$. By Lemma 25, we can
reach every other interval of type \((v, n)\). If \(v \geq 2\), then by Lemma 26 we can reach an interval of type \((v-1, n)\). Thus we can reach all intervals \([V, Q]\), one for each non-empty subset \(V\) of \(Q\). Since there are at most \(2^n - 1\) collections and that many can be reached, no other collection can be reached.

For \(A_0\), the initial state of \(A^Q_0\) is the empty collection, which is the interval \([0, 0]\).

Now suppose we have reached an interval of type \((0, u)\). By Lemma 25, we can reach every other interval of type \((0, u)\). If \(u \leq n - 2\), then by Lemma 26 we can reach an interval of type \((0, u + 1)\). Thus we can reach all intervals \([0, U]\), one for each non-empty subset \(U\) of \(Q\). Since there are at most \(2^n - 1\) collections and that many can be reached, no other collection can be reached. Hence the proposition holds. 

**Proposition 28 (Tightness)** For \(1 \leq r \leq n - 1\), the quotient complexity of any atom of \(L_n\) with \(r\) complemented quotients is \(f(n, r)\).

**Proof.** Let \(A_P\) be an atom of \(L_n\) with \(n - r\) uncomplemented quotients, where \(1 \leq r \leq n - 1\), that is, let \(P\) be the set of subscripts of the uncomplemented quotients. Let \(A_P^\prime\) be the modified automaton with the initial state \(P\). As discussed above, \(A^Q_{P^\prime}\) is minimal; hence it suffices to prove the reachability of \(f(n, r)\) collections.

We start with the interval \([P, P]\) of type \((n-r, n-r)\). By Lemmas 25 and 26, we can now reach all intervals of types

\[
(n-r, n-r), (n-r-1, n-r), \ldots, (1, n-r),
(n-r, n-r+1), (n-r-1, n-r+1), \ldots, (1, n-r+1),
\]

\[
\ldots
(n-r, n-1), (n-r-1, n-1), \ldots, (1, n-1).
\]

Since the number of intervals of type \((v, u)\) is \(\binom{n}{u}\binom{n}{v}\), we can reach

\[
g(n, r) = \sum_{v=n-r+1}^{n-1} \sum_{u=1}^{r} \binom{n}{u} \binom{n}{v}
\]

intervals. Changing the first summation index to \(k = n-u\), we get

\[
g(n, r) = \sum_{k=1}^{r} \sum_{v=1}^{n-r} \binom{n}{k} \binom{n-k}{v}.
\]

Note that \(\binom{n}{k} \binom{n-k}{v} = \binom{n}{k+v} \binom{k+v}{k}\), because

\[
\binom{n}{k} \binom{n-k}{v} = \frac{n!}{(n-k)!k!}(n-k)! = \frac{n!}{k!(n-k-v)!}, \quad \text{and}
\]

\[
\binom{n}{k+v} \binom{k+v}{k} = \frac{n!}{(k+v)!(n-k-v)!}(k+v)! = \frac{n!}{(n-k-v)!(k+v)!}.
\]
Now, we can write $g(n, r) = \sum_{k=1}^{r} \sum_{v=1}^{n-r} \binom{n}{k+v} \binom{k+v}{k}$, and changing the second summation index to $h = k + v$, we have

$$g(n, r) = \sum_{k=1}^{r} \sum_{h=k+1}^{n-r} \binom{n}{h} \binom{h}{k}.$$

We notice that $g(n, r) = f(n, r) - 1$. From the interval $[V, V]$, where $V = \{0, 1, \ldots, n - r - 1\}$, we reach the empty quotient by input $c$, since $V$ contains 0, but not $n - 1$. Since we can reach $f(n, r)$ intervals, no other collection can be reached, and the proposition holds.

9. Conclusions

The atoms of a regular language $L$ are its building blocks. We characterized atomic NFAs of $L$. We studied the quotient complexity of the atoms of $L$ as a function of the quotient complexity of $L$. We computed an upper bound for the quotient complexity of any atom and exhibited languages $\{L_n\}$ whose atoms meet this bound.

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