A Characterization of de Morgan Algebras *

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Abstract

In this note we show that every de Morgan algebra is isomorphic to a *two-subset algebra*, $\langle P, \sqcup, \sqcap, \sim, 0_P, 1_P \rangle$, where P is a set of pairs (X, Y) of subsets of a set I, $(X, Y) \sqcup (X', Y') = (X \cap X', Y \cup Y')$, $(X, Y) \sqcap (X', Y') = (X \cup X', Y \cap Y')$, $\sim (X, Y) = (Y, X)$, $1_P = (\emptyset, I)$, and $0_P = (I, \emptyset)$. This characterization generalizes a previous result that applied only to a special type of de Morgan algebras called ternary algebras.

1 Introduction

In 1935 Moisil [7] studied *de Morgan lattices*, which are distributive lattices with 0 and 1, and a unary operation $^-$ that satisfies $\overline{\overline{x}} = x$ and de Morgan's

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laws. In 1957 Białynicki-Birula and Rasiowa studied these lattices under the name *quasi-Boolean algebra* [2], and used the term *quasi-complement* for the unary operation. They showed that every quasi-Boolean algebra is isomorphic to a *quasi-field* of sets, which is a ring of sets with a complicated quasi-complementation that is *not* set complementation in general. The construction uses the well-known theorem of Birkhoff that every distributive lattice is isomorphic to a ring of sets (see, for example [3]: Corollary 2, p. 194, [1]: Theorem 3, p. 63, [8]: 2.3 p. 44.) Quasi-Boolean algebras were also discussed in [8], and in [1] under the name *de Morgan algebras*. We use the latter term for these objects. Thus, a de Morgan algebra is an algebra $\langle S, +, *, ^{-}, 0, 1\rangle$, where $\langle S, +, *, 0, 1\rangle$ is a distributive lattice with 0 and 1, and $^{-}$ is a unary operation, called *quasi-complement*, that satisfies $\overline{\overline{x}} = x$ and de Morgan's laws.

A *ternary algebra* is a de Morgan algebra with an additional constant Φ satisfying $\Phi = \overline{\Phi}$, $(x + \overline{x}) + \Phi = x + \overline{x}$, and $(x * \overline{x}) * \Phi = x * \overline{x}$. Ternary algebras have applications in computer science; see, for example [5]. Brzozowski, Lou, and Negulescu [4] characterized finite ternary algebras in terms of subset-pair algebras defined below. This characterization was extended to the infinite case by Ésik, who proved the result by also using the representation of distributive lattices by rings of sets [6]. With a slight modification of the ideas of [6], we obtain a characterization of de Morgan algebras.

2 Two-Subset Algebras

Let I be a non-empty set, and let P be a set of ordered pairs (X, Y) of subsets of I. We call P a *two-subset algebra* if it is closed under the operations $\sqcup$, $\sqcap$, and $\sim$ defined below, and if $1_P = (\emptyset, I)$ and $0_P = (I, \emptyset)$ are in P.

$$(X, Y) \sqcup (X', Y') = (X \cap X', Y \cup Y'),$$

$$(X, Y) \sqcap (X', Y') = (X \cup X', Y \cap Y'),$$

$$\sim (X,Y) = (Y,X).$$

A restricted version of two-subset algebra was introduced in [4] under the name of *subset-pair algebra*. There, it was also required that the subsets satisfy $X \cup Y = I$ and that there be a third constant (I, I) in P.

The following is easily verified:

Proposition. *Every two-subset algebra is a de Morgan algebra.*

3 A Characterization of de Morgan Algebras

Given a de Morgan algebra $\langle S, +, *, \bar{}, 0, 1\rangle$, we construct a two-subset algebra $\langle P, \sqcup, \sqcap, \sim, 0_P, 1_P\rangle$ as follows.

Since S is a distributive lattice, it can be embedded in a ring of sets. Thus there exists a set I such that $\langle S, +, *, 0, 1\rangle$ is isomorphic to $\langle X, \cap, \cup, I, \emptyset\rangle$, where X is a set of subsets of I. Here $+$ corresponds to set intersection and $*$, to set union. If $s \in S$, let $\sigma(s)$ denote the set assigned to s by the embedding. In general, X is not closed under set complementation.

In a de Morgan algebra, the assignment $x \mapsto \overline{x}$ satisfies $x \le y$ iff $\overline{x} \ge \overline{y}$, where $\le$ is the usual partial order defined by $x \le y$ iff $x + y = y$ iff $x * y = x$. Thus the assignment $x \mapsto \overline{x}$ is an antiautomorphism of S. Note that $x + y = z$ iff $\overline{x} * \overline{y} = \overline{z}$ and $x * y = z$ iff $\overline{x} + \overline{y} = \overline{z}$, because of de Morgan's laws. Given an embedding of a de Morgan algebra into a ring of sets, we now define a second embedding by associating with $s \in S$ the set $\sigma(\overline{s})$. Now $\langle S, +, *, 0, 1\rangle$ is isomorphic to $\langle X, \cup, \cap, \emptyset, I\rangle$. Here $+$ corresponds to set union and $*$, to intersection.

It follows that we can also use a double embedding for any de Morgan algebra, where each element $s \in S$ corresponds to an ordered pair $(\sigma(s), \sigma(\overline{s}))$ of subsets of I. We have the following properties of the double embedding. For all $x, y, z \in S$,

$$x + y = z \text{ iff } \sigma(x) \cap \sigma(y) = \sigma(z), \quad x * y = z \text{ iff } \sigma(x) \cup \sigma(y) = \sigma(z),$$
$$x + y = z \text{ iff } \sigma(\overline{x}) \cup \sigma(\overline{y}) = \sigma(\overline{z}), \quad x * y = z \text{ iff } \sigma(\overline{x}) \cap \sigma(\overline{y}) = \sigma(\overline{z}),$$

$$\overline{x} = y \text{ iff } (\sigma(x), \sigma(\overline{x})) = (\sigma(\overline{y}), \sigma(y)).$$

The following is now evident:

Theorem. *Every de Morgan algebra is isomorphic to a two-subset algebra.*

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