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A characterization of signed hypergraphs and its applications to VLSI via minimization and logic synthesis

C.-J. Shi [a,*], J.A. Brzozowski [b]

[a] Department of Electrical Engineering, University of Washington, Seattle, WA 98195, USA

[b] Department of Computer Science, University of Waterloo, Waterloo, Ont. Canada N2L 3G1

Abstract

A recently introduced graph-theoretic notion of signed hypergraph is studied. In particular, a structural characterization of balanced signed hypergraphs is given, and two optimization problems related to the balance property – the maximum balance and the minimum covering problems – are introduced and characterized. It is shown that both problems are NP-complete in general. Applications of the theory of signed hypergraphs to two VLSI optimization problems, namely via minimization and constrained logic encoding, are described. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: VLSI; Via minimization; Logic minimization; Signed hypergraph; Balance; Partitioning

1. Introduction

A *graph* consists of a set of *vertices* and a set of *edges*, each edge being incident with two vertices. A *hypergraph* is a generalization of a graph in which each edge may be incident with any number of vertices. A *signed hypergraph* is a generalization of a hypergraph in which each vertex-edge incidence is assigned either +1 (in which case it is considered to be a positive incidence) or −1 (a negative incidence).

The notion of signed hypergraph was introduced recently to model the constrained via minimization (CVM) problem of two-layer routings [19]. In the integrated circuit (IC) and printed circuit board (PCB) design, the use of a large number of vias (contacts) between conducting layers, affects adversely both system performance and reliability. The constrained via minimization problem is to minimize the number of vias by appropriately assigning layers to wire segments. It has long been observed that the CVM problem has a graph-like structure, and considerable effort has been devoted to the application of conventional graph theory – with various degrees of

* Corresponding author. E-mail: cjshi@ee.washington.edu.

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success [4, 5, 11, 15, 17, 26]. A survey of the previous work in this direction can be found in a recent article by Joy and Ciesielski [14]. However, the problem has not been successfully formulated until the introduction of signed hypergraphs.

The formulation of the CVM problem in the framework of signed hypergraphs is based on the concept of *edge balance*. An edge e is said to be *balanced* by a bipartition of the set of vertices if all the vertices that are positively incident with e are in one block of the bipartition and all the vertices that are negatively incident with e are in the other block. The CVM problem is then formulated as the problem of finding a bipartition that balances as many edges as possible. This is called the *maximum balance* problem in a signed hypergraph. It is the first complete, yet simple, graph-theoretic model for constrained via minimization. In this model, solutions of several CVM benchmark routings, which were quite involved previously, turn out to be straightforward [20]. Furthermore, it has been demonstrated that signed-hypergraph-based heuristics for CVM are more robust and efficient than graph-based heuristics [20]. This parallels the observation on the performances of hypergraph-based and graph-based heuristics for netlist partitioning [16].

In this paper, we study some structural properties of signed hypergraphs. A signed hypergraph is said to be *balanced* if there exists a bipartition of its vertices that balances all of its edges. We define the *sign* of a cycle in a signed hypergraph to be the sign of the product of all the vertex-edge incidences involved in the cycle. We prove that a signed hypergraph is balanced if and only if it is free of negative cycles. Our proof also gives a linear-time algorithm for testing whether a signed hypergraph is balanced. By using the structural characterization mentioned above, we prove that the maximum balance problem is NP-complete, even in the planar case.

Next, we observe that a variation of the maximum balance problem provides a precise graph-theoretic formulation for constrained encoding – a fundamental problem in VLSI logic synthesis. It has been established that (a) race-free state assignment for asynchronous sequential machines, (b) delay-free state assignment for asynchronous sequential machines without essential hazards, (c) optimum state assignment for synchronous sequential machines, and (d) PLA decomposition all reduce to the constrained logic encoding problem [22]. The abstract formulation corresponding to all these problems is the so-called *minimum covering* problem: Given a signed hypergraph, find a minimum number of vertex bipartitions such that each edge is balanced by at least one bipartition. In this paper, we prove that the minimum covering problem is NP-complete.

The results described in this paper are a part of the first author's Ph. D. dissertation [21]. Our new contribution includes the characterization of signed hypergraphs (Theorems 3.1 and 3.2), the complexity study of the two optimization problems (Theorems 4.1, 4.2 and 5.1), and the formulation of the constrained encoding problem. The formulation of the constrained via minimization as the maximum balance problem appeared originally in [19, 20, 23], and is included here for completeness. The paper is structured as follows: Section 2 formally introduces the notion of signed hypergraph and describes the basic terminology and notation. Section 3 presents a structural characterization of balanced signed hypergraphs. Section 4 establishes the NP-completeness of the maximum balance problem. Section 5 proves the NP-completeness of the minimum covering problem. Sections 6 and 7 describe the applications of signed hypergraph theory to the constrained via minimization problem and the constrained encoding problem, respectively. Section 8 discusses some mathematical concepts related to the notion of signed hypergraph. Section 9 concludes the paper.

2. Terminology and notation

A *signed hypergraph* H is an ordered triple[1] $(V(H), E(H), \psi_H)$ – or simply (V, E, ψ), if H is understood – consisting of a set V of *vertices*, a set E, disjoint from V, of *edges*, and an *incidence function* $\psi : V \times E \to \{-1, 0, 1\}$.

The *incidence matrix* of a signed hypergraph $H = (V, E, \psi)$ is a $|V| \times |E|$ matrix

$$\boldsymbol{\psi} = (\psi_{ij})$$

where $\psi_{ij} = \psi(v_i, e_j)$, and $|V|$ ($|E|$) denotes the number of vertices (edges) of H. Clearly, each signed hypergraph has a unique incidence matrix, and each $(0, \pm 1)$-matrix corresponds to an incidence matrix of a signed hypergraph.

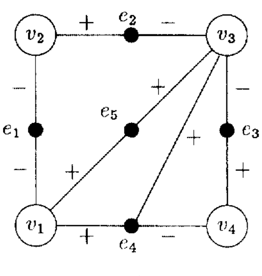
The *dual* $H^* = (V^*, E^*, \psi^*)$ of a signed hypergraph $H = (V, E, \psi)$ is a signed hypergraph where $V^* = E$, $E^* = V$, and ψ^* is the transpose of ψ. Consequently, $(H^*)^* = H$.

If e is an edge and v a vertex such that $\psi(v, e) \neq 0$, then v is said to be *incident with* e and vice versa. More specifically, if $\psi(v, e) = 1$, v is *positively* incident with e; if $\psi(v, e) = -1$, v is *negatively* incident with e. The incidence function permits an edge and a vertex to meet only once. Thus no edge can connect a vertex to itself; i.e., "self-loops" are not allowed in signed hypergraphs. Two or more vertices are said to be *adjacent* if they are incident with the same edge. The *degree* $d(v)$ of a vertex v in H is the number of edges of H incident with v. We denote by $\triangle_V$ the maximum degree of the vertices of H. The *degree* $d(e)$ of an edge e in H is the number of vertices of H incident with e. We denote by $\triangle_E$ the maximum degree of the edges of H.

A signed hypergraph H degenerates to a *signed graph* if $d(e) = 2$ for every edge $e \in E$. It degenerates to a *hypergraph* if $\psi(v, e) \in \{0, 1\}$ for all $v \in V$ and $e \in E$. Finally, a signed hypergraph H degenerates to a *graph* if $\psi(v, e) \in \{0, 1\}$ and $d(e) = 2$, for all $v \in V$ and $e \in E$. We usually denote a (signed) graph by G, and write $V(G), E(G)$, and ψ_G.

A signed hypergraph is shown in Fig. 1, where a circle represents a vertex, and a small solid circle (called *edge node*) with several line segments attached to it represents an edge. The graph obtained by treating edge nodes in the same way as the vertices of H is called the *underlying graph* of H. Figs. 2 and 3 are two more examples of

[1] We follow here the notation of Bondy and Murty [3] because the use of triples provides a simple way of specifying signed hypergraphs.

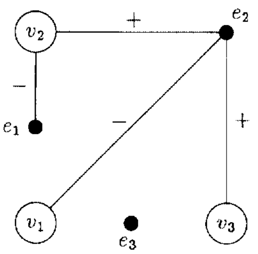


(a)

$$\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} \end{array}$$

(b)

Fig. 1. (a) A signed hypergraph H; (b) its incidence matrix.

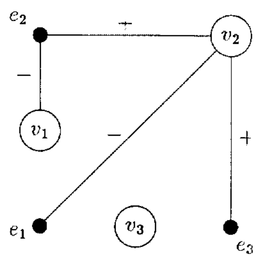


(a)

$$\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \end{array}$$

(b)

Fig. 2. A signed hypergraph H_1; (b) its incidence matrix.



(a)

$$\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \end{array}$$

(b)

Fig. 3. A signed hypergraph H_2; (b) its incidence matrix.

signed hypergraphs, where the signed hypergraph in Fig. 3 is the dual of the signed hypergraph in Fig. 2. Edges e_1 and e_3 in Fig. 2 are included for completeness. We note that they arise naturally in the formulation of practical application problems (Sections 6 and 7).

The underlying graph of a signed hypergraph has the property that all the vertices are partitioned into two subsets (the set of hypergraph vertices and the set of edge nodes) such that every edge connects one vertex from each subset; such a graph is called *bipartite*. Each signed hypergraph has a unique underlying bipartite graph, and each bipartite graph corresponds to the underlying graph of a signed hypergraph. Thus we may use three terminologies – signed hypergraphs, $(0, \pm 1)$ matrices, and ± 1 (edge-) weighted bipartite graphs – interchangeably.

A graph is said to be *planar* if it is possible to embed it in a plane so that no two edges intersect. A signed hypergraph is *planar* if its underlying graph is planar. For example, the signed hypergraphs in Figs. 1–3 are planar.

A signed hypergraph H' is a *subhypergraph* of H (written $H' \subseteq H$) if $V(H') \subseteq V(H)$, $E(H') \subseteq E(H)$, and $\psi_{H'}$ is the restriction of $\psi(H)$ to $V(H') \times E(H')$. For example, H' defined by $V(H') = \{v_1, v_2, v_3\}$, $E(H') = \{e_1, e_3, e_4\}$, and

$$\psi_{H'} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

is a subhypergraph of H defined in Fig. 1. Note that, in terms of the incidence matrix, the restriction of ψ_H to $V(H') \times E(H')$ is obtained by crossing out those rows of ψ_H that are not in $V(H')$ and those columns of ψ_H that are not in $E(H')$. A *(vertex-)spanning subhypergraph* H' of H is a subhypergraph with $V(H') = V(H)$.

Suppose that V' is a nonempty subset of V. A *subhypergraph of* $H = (V, E, \psi)$ *induced by vertex set* V' is the subhypergraph whose vertex set is V' and whose edge set is the set of those edges of H that are incident *only* with vertices in V'; it is denoted by $H(V')$. We also say that $H(V')$ is a *vertex-induced subhypergraph* of H. For example, the subhypergraph of H in Fig. 1 induced by the vertex set $V' = \{v_1, v_2, v_3\}$ is the triple (V', E', ψ'), where $E' = \{e_1, e_2, e_5\}$ and

$$\psi' = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Let V'' be a nonempty subset of V. Then $H(V - V'')$ is a subhypergraph induced by vertex set $V - V''$, i.e., the subhypergraph obtained from H by deleting the vertices in V'' together with all their incident edges.

Suppose that E' is a nonempty subset of E. A *subhypergraph of* $H = (V, E, \psi)$ *induced by edge set* E' is the subhypergraph whose vertex set is the set of vertices incident with edges in E' and whose edge set E'; it is denoted by $H(E')$. We also say that $H(E')$ is an *edge-induced subhypergraph* of H. Let E'' be a nonempty subset of E. The spanning subhypergraph with edge set $E - E''$, written as $H(E - E'')$, is the subhypergraph obtained from H by deleting the edges in E''. For example, the subhypergraph $H(E - \{e_4\})$ of H in Fig. 1 is the triple (V'', E'', ψ'') defined below:

$$V'' = \{v_1, v_2, v_3, v_4\},$$

$$E'' = \{e_1, e_2, e_3, e_5\},$$

$$\psi'' = \begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that the definition of edge-induced subhypergraph is *not* the dual of the definition of vertex-induced subhypergraph.

Let $q \geqslant 1$ be an integer. A *path of length* q is defined to be a sequence

$$(v_1, e_1, v_2, e_2, \ldots, v_q, e_q, v_{q+1})$$

such that

$$\forall i\colon\ 1 \leqslant i \leqslant q\colon\quad v_i \neq v_{i+1},\ \psi(v_i, e_i) \neq 0 \quad \text{and} \quad \psi(v_{i-1}, e_i) \neq 0.$$

The *sign* of a path is equal to the sign of

$$\prod_{i=1}^{q} \psi(v_i, e_i)\psi(v_{i+1}, e_i).$$

A *positive* (*negative*) *path* is a path with positive (negative) sign. A *cycle* is a path in which $v_{q+1} = v_1$; note that the length of any cycle is necessarily greater than 1. An *odd* (*even*) *cycle* is a cycle with odd (even) length. For example, for the signed hypergraph of Fig. 1, the path $(v_1, e_1, v_2, e_2, v_3)$ is negative. Cycles $(v_1, e_1, v_2, e_2, v_3, e_5, v_1)$ and $(v_1, e_4, v_3, e_2, v_2, e_1, v_1)$ are negative, whereas the cycle

$$(v_1, e_1, v_2, e_2, v_3, e_5, v_1, e_4, v_3, e_2, v_2, e_1, v_1)$$

is positive.

A *bipartition* π of H is a separation of V into a pair of subsets, say (V_1, V_2), such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. Here V_1 and V_2 are called *blocks* of π. The block containing vertex v will be denoted by $\pi(v)$. An edge e is *balanced* by a bipartition π if the following condition is satisfied:

$$\forall v_1, v_2 \in V,\quad \psi(v_1, e) = \psi(v_2, e) \neq 0 \quad \text{implies } \pi(v_1) = \pi(v_2), \tag{1}$$

$$\psi(v_1, e) = -\psi(v_2, e) \neq 0 \quad \text{implies } \pi(v_1) = \pi(v_2). \tag{2}$$

For example, in Fig. 1, bipartition $(\{v_1, v_2\}, \{v_3, v_4\})$ balances edges e_1 and e_2, but not edges e_3, e_4 and e_5; bipartition $(\{v_1, v_2, v_3\}, \{v_4\})$ balances edges e_1, e_3, e_4 and e_5, but not edge e_2.

A bipartition is said to *balance* a signed hypergraph if it balances all of its edges. A signed hypergraph is said to be *balanced* if there exists a bipartition that balances all the edges. For example, bipartition $(\{v_1, v_2, v_3\}, \{v_4\})$ balances the subhypergraph $H(E - \{e_2\})$ of H in Fig. 1; hence this subhypergraph is balanced.

3. Fundamental characterizations

In this section, we present some characterizations of the balance property of signed hypergraphs.

Proposition 3.1. *Every subhypergraph of a balanced signed hypergraph is balanced.*

Proof. Every edge of the subhypergraph is balanced by the bipartition that balances the original signed hypergraph. □

Theorem 3.1 (Structure theorem). *A signed hypergraph is balanced if and only if it is free of negative cycles.*

Proof. We first prove the only if part. Suppose that a signed hypergraph H is balanced; then there exists a bipartition π such that, for each edge $e \in E(H)$, $\psi(v_1,e)=\psi(v_2,e)\neq 0$ implies that v_1 and v_2 are in the same block of π, and $\psi(v_1,e)=-\psi(v_2,e)\neq 0$ implies that v_1 and v_2 are in different blocks. We say that a paths is *cut* by π if there exists a path segment of the form $v_ie_iv_{i+1}$ such that v_i and v_{i+1} belong to different blocks of π. Then the sign of any cycle in H may be determined by counting the number of times the cycle is cut by π. Since a cycle can only be cut by a bipartition an even number of times, every cycle must be positive.

We prove the if part by construction. Suppose that we are given a signed hypergraph H that is free of negative cycles. We construct a bipartition $\pi=(X,Y)$ as follows: Arbitrarily select a vertex v and assign it to the block X. Since H is free of negative cycles, for each pair of vertices v and v', all paths joining v and v' have the same sign. Search H in a depth-first manner and calculate the sign of any path from v to v'. If the sign is positive, then v' is in X, otherwise v' is in Y.

We claim that all the edges in H are balanced by π. By construction, for each edge $e\in E(H)$ which connects vertices v_1 and v_2, $\psi(v_1,e)=\psi(v_2,e)\neq 0$ implies that the path joining v_1 and v_2 is positive, and thus that v_1 and v_2 are in the same block of π. If $\psi(v_1,e)=-\psi(v_2,e)\neq 0$, then the path joining v_1 and v_2 is negative, and v_1 and v_2 are in different blocks of π. Therefore edge e is balanced by π, and H is a balanced signed hypergraph. □

A slight variation of the construction used in the proof above gives an algorithm that checks whether a signed hypergraph is balanced in linear time in the size of H, i.e., the number of non-zeros in its incidence matrix. We state this observation as a corollary.

Corollary 3.1. *Checking the balance of a signed hypergraph $H=(V,E,\psi)$ takes linear time in the size of H.*

Theorem 3.2 (Duality theorem). *The dual of a balanced signed hypergraph is balanced.*

Proof. By Theorem 3.1, a signed hypergraph is balanced if and only if it is free of negative cycles. Since a signed hypergraph and its dual have the same underlying bipartite graph, cycles in a signed hypergraph have a one-to-one correspondence with cycles in its dual. Thus the dual of a balanced signed hypergraph is also free of negative cycles, and it is therefore balanced. □

4. The maximum balance problem and its complexity

In this section, we study the following problem: Given a signed hypergraph, find a bipartition such that the number of balanced edges is maximized. This is called the *maximum balance problem.*

The main result of this section is the NP-completeness of the maximum balance problem; more precisely, we establish a clear boundary between polynomial-time solvable cases and the general NP-complete case. In addition, the complexity of the dual of the maximum balance problem and the complexity of the maximum balance problem in a *planar* signed hypergraph are investigated.

Before we prove the NP-completeness results, we first make the following observations. For a signed hypergraph H, the maximum balance problem is that of finding a bipartition π such that the number of balanced edges is maximized. From the definition of balance, the spanning subhypergraph of H with the set of edges balanced by π is balanced. In other words, after the set E' of edges that are not balanced by π is removed, the remaining signed hypergraph $H(E - E')$ is balanced. By Theorem 3.1, $H(E-E')$ is free of negative cycles. By Corollary 3.1, the construction for $H(E-E')$ of a bipartition π that balances all the edges in $E-E'$ takes linear time. Therefore, an alternative formulation of the maximum balance problem, which is of the same time complexity as the original, is as follows:

> Given a signed hypergraph $H=(V,E,\psi)$, find a set $E' \subseteq E$ with minimum cardinality such that $H(E - E')$ is free of negative cycles.

Following the terminology of Yannakakis [27], we call this the *edge-deletion balance* problem for a signed hypergraph. The corresponding decision problem is: Given H and an integer $k \geqslant 0$, does there exist a set E' of k edges such that $H(E - E')$ is free of negative cycles?

A key to our proof of the NP-completenesses of the maximum balance problem and its variants is provided by Lemmas 4.1 and 4.2, which state some properties of the following construction: Given a graph $G=(V,E,\psi)$, construct a signed hypergraph H with the same set of vertices and the same set of edges. For each pair of vertices v_i and v_j in G joined by an edge e in E, set $\psi_H(v_i,e)=1$ and $\psi_H(v_j,e)=-1$ in H. (The opposite assignment can also be chosen; it is not important for our purposes which incidence is considered positive.) We refer to this as the *sign construction.* The sign construction takes linear time. Since $d(e)=2$ for every $e \in E$, the signed hypergraph

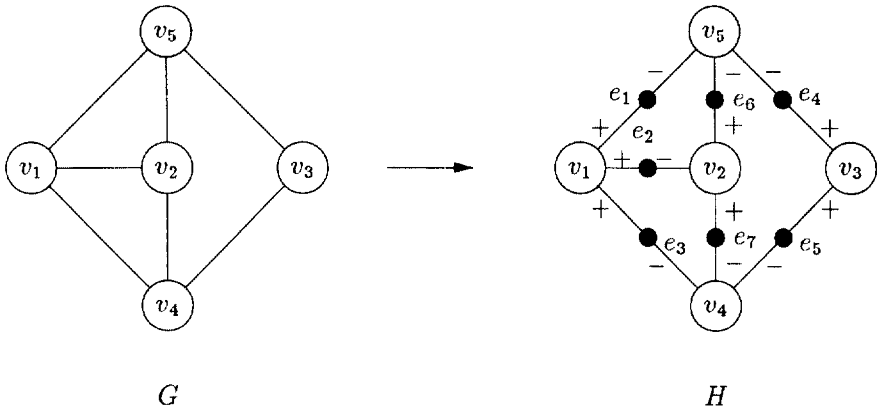


Fig. 4. An illustration of the sign construction.

H so constructed is in fact a signed graph; i.e., $\triangle_E = 2$. An example of the sign construction is given in Fig. 4.

Lemma 4.1. *Let G be a graph, and H any signed hypergraph obtained from G by the sign construction. Then there exists a set E' of k edges in G such that $G(E-E')$ is bipartite if and only if there exists a set of k edges in H such that $H(E-E')$ is free of negative cycles.*

Proof. Let e be an arbitrary edge in H, and let v_1 and v_2 be two vertices incident with e. By the construction of H, we always have $\psi(v_1,e)\psi(v_2,e) = -1$. Therefore the sign of any cycle in H is determined by the number of edges in the cycle: It is positive if the cycle is even, and it is negative otherwise. Thus H is free of negative cycles, if and only if G is free of odd cycles. It is well known that G is free of odd cycles if and only if G is bipartite [1]. □

Similarly, we have the following result:

Lemma 4.2. *Let G be a graph, and H any signed hypergraph obtained from G by the sign construction. Then there exists a set V' of k vertices in G such that $G(V-V')$ is bipartite if and only if there exists a set of k vertices in H such that $H(V-V')$ is free of negative cycles.*

Theorem 4.1. *The edge-deletion balance problem for a signed hypergraph $H=(V,E,\psi)$ is NP-complete even if* (a) *$\triangle_E = 2$ and $\triangle_V = 3$, or* (b) *$\triangle_E = 3$ and $\triangle_V = 2$.*

Proof. We first show that the edge-deletion balance problem is in NP. Given a set E' of k edges, checking whether $H(E-E')$ is free of negative cycles is equivalent to checking whether a bipartition exists that balances all the edges in $E-E'$; this can be done in linear time by Corollary 3.1.

The NP-hardness of the edge-deletion balance problem for signed hypergraphs with $\triangle_E = 2$ and $\triangle_V = 3$ is proved by showing that the edge-deletion bipartite problem can be reduced in polynomial time to our problem. The edge-deletion bipartite problem, which was proved to be NP-complete by Garey et al. [9], and Yannakakis [27], is described below:

Edge-deletion bipartite problem

Instance: A graph $G=(V,E,\psi)$ with $d(v)=3$ for every vertex $v \in V$, and an integer k.

Question: Does there exist a set E' of k edges such that $G(E-E')$ is bipartite?

Given a graph $G=(V,E,\psi)$ with $d(v)=3$ for every vertex $v \in V$, we use the sign construction to generate a signed hypergraph H. Since $d(v)=3$ for every $v \in V$ in G, the signed hypergraph H so constructed has $\triangle_E = 2$ and $\triangle_V = 3$. By Lemma 4.1, there exists a set E' of k edges in G such that $G(E-E')$ is bipartite if and only if there exists a set of k edges in H such that $H(E-E')$ is free of negative cycles. Therefore the problem for a signed hypergraph with $\triangle_E = 2$ and $\triangle_V = 3$ is NP-hard.

To prove the NP-hardness of the maximum balance problem for a signed hypergraph with $\triangle_E = 3$ and $\triangle_V = 2$, we use the dual of the edge-deletion bipartite problem – the vertex-deletion bipartite problem, which was proved to be NP-complete by Choi, Nakajima and Rim [5]:

Vertex-deletion bipartite problem

Instance: A graph $G=(V,E,\psi)$ with $d(v)=3$ for every vertex $v \in V$, and an integer k.

Question: Does there exist a set V' of k vertices such that $G(V-V')$ is bipartite?

Given a graph with $d(v)=3$ for every vertex $v \in V$, we apply the sign construction to generate a signed hypergraph $(V(H),E(H),\psi_H)$. We then construct the dual H^* of H. Since $d(v)=3$ for every vertex $v \in V$, H^* is such that $\triangle_V = 2$ and $\triangle_E = 3$.

We claim that there exists a set V' of k vertices in G such that $G(V-V')$ is bipartite if and only if there exists a set E' of k edges in H^* such that $H^*(E-E')$ is balanced. By Lemma (4.2), there exists a set V' of k vertices in G such that $G(V-V')$ is bipartite if and only if there exists a set V' of k edges in H such that $H(V-V')$ is balanced. (See Fig. 4.) By Theorem (3.2), there exists a set V' of k vertices in H such that $H(V-V')$ is balanced, if and only if there exists a set E' of k edges in H^* such that $H^*(E-E')$ is balanced. (See Fig. 5.) Thus the claim is indeed true. Both the sign construction and the dual construction take linear time. □

The dual of the edge-deletion balance problem is the *vertex-deletion balance* problem:

Given a signed hypergraph $H=(V,E,\psi)$, find a set $V' \subset V$ with minimum cardinality such that $H(V-V')$ is balanced.

As a corollary of Theorems 3.2 and 4.1, we have the following result.

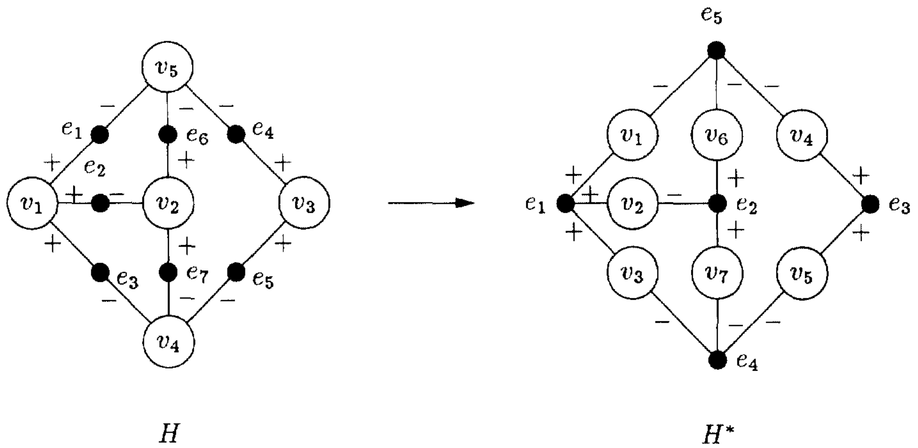


Fig. 5. An example of the dual construction.

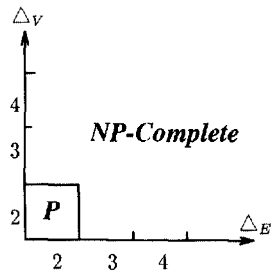


Fig. 6. Complexity of the maximum balance problem.

Corollary 4.1. *The vextex-deletion balance problem for a signed hypergraph is NP-complete even if* (a) $\triangle_E = 2$ *and* $\triangle_V = 3$, *or* (b) $\triangle_E = 3$ *and* $\triangle_V = 2$.

To complete our analysis, we consider the case with $\triangle_E = 2$ and $\triangle_V = 2$. For a general signed hypergraph, it is easy to see that this case can be solved in linear time. This leads to a complete understanding of the complexity of the maximum balance problem in a general signed hypergraph, as illustrated in Fig. 6.

In the rest of this section, we consider the effect of planarity on the complexity of the maximum balance problem. This is motivated by the fact that the layer assignment problem of integrated circuit layout gives rise to signed hypergraphs that are usually planar.

According to Choi et al. [5], the vertex-deletion bipartite problem when restricted to a planar graph is still NP-complete when $\triangle_V > 3$. By using the same constructions and arguments as in the proof of Theorem 4.1, and nothing that the dual of a planar signed hypergraph is planar, we have the following result:

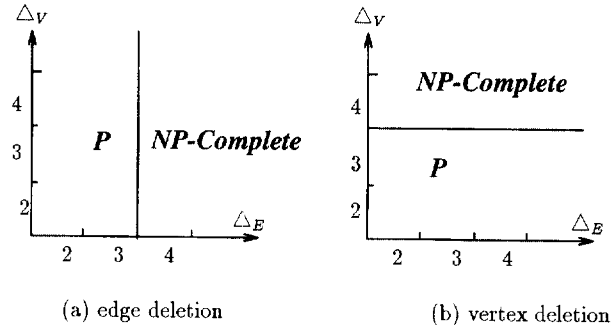


Fig. 7. Complexity of a planar maximum balance.

Theorem 4.2. *The edge-deletion balance problem is NP-complete for a planar signed hypergraph with* $\triangle_E > 3$.

Corollary 4.2. *The vertex-deletion balance problem is NP-complete for a planar signed hypergraph with* $\triangle_V > 3$.

For the maximum balance problem of planar signed hypergraphs with $\triangle_E \leqslant 3$, we have developed polynomial-time algorithms in [21]. A complete picture of the complexity results for maximum balance in planar signed hypergraphs is in Fig. 7.

5. The minimum covering problem and its complexity

In this section, we consider the following problem: Given a signed hypergraph, find a minimum number of bipartitions such that each edge is balanced by at least one bipartition. This is called the *minimum covering problem.* The main result of this section is the NP-completeness of the minimum covering problem.

We introduce a notion of set decomposition. If E is a set, a *decomposition* of E is a set $\{E_i \mid i = 1,\dots,k\}$ of subsets E_i of E such that $\bigcup_{i=1}^{k} E_i = E$. If E is a set of edges of a signed hypergraph H, and $\{E_i \mid i = 1,\dots,k\}$ is a decomposition of E, then $\{H(E_i) \mid i = 1,\dots,k\}$ is also called a *decomposition* of H, where $H(E_i)$ is the subhypergraph induced by E_i for $i = 1,\dots,k$.

By Theorem 3.1, the subhypergraph induced by the set of edges that are balanced by a bipartition is balanced, i.e., free of negative cycles. Thus an alternative formulation of the minimum covering problem is as follows:

> Given a signed hypergraph $H = (V, E, \psi)$, find a decomposition $\{H_i \mid i = 1,\dots,k\}$ of H such that (a) H_i is balanced and (b) k is minimized.

This is the *balanced subhypergraph decomposition problem.* If $k = 1$, the problem degenerates to testing whether a signed hypergraph H is balanced, which is solvable in linear time. If $k = |E|$, the problem can also be solved in linear time by simply choosing each edge as a signed hypergraph.

Theorem 5.1. *The balanced subhypergraph decomposition problem for signed hypergraphs is NP-complete.*

Proof. We first show that the balanced subhypergraph decomposition problem is in NP. Given a signed hypergraph H and a set $\{H_i \mid i = 1, \ldots, k\}$, we can verify in time polynomial in the size of the problem, i.e., in the size of H plus the sum of the sizes of the H_i, $i = 1, \ldots, k$, whether H_i is balanced for all i and whether $\{H_i \mid i = 1, \ldots, k\}$ is indeed a decomposition of H.

To prove the NP-hardness, we show that the graph K-colorability problem is polynomial-time reducible to our problem. The graph K-colorability problem described below is a known NP-complete problem [8].

Graph K-colorability problem
Instance: A graph $G = (V, E, \psi)$, and an integer $2 < K < |V|$.
Question: Does there exist a mapping $f : V \to \{1, \ldots, K\}$ such that, for each edge e incident with vertices v_i and v_j, $f(v_i) \neq f(v_j)$?

If K is restricted so that $K = 2^k$, $1 < k < \log_2 |V|$, this is the graph 2^k-colorability problem. Since the graph K-colorability problem remains NP-complete for any fixed $K > 2$ – for example, the graph 3-colorability problem is NP-complete – the graph 2^k-colorability problem is also NP-complete.

To show that the graph 2^k-colorability problem is polynomial-time reducible to our problem, we first observe that the following problem

Bipartite subgraph decomposition problem
Instance: A graph $G = (V, E, \psi)$ and an integer k, $1 < k < \log_2 |V|$.
Question: Does there exist a decomposition $\{G_i \mid i = 1, \ldots, k\}$ of G such that G_i is bipartite for all i?

can be transformed into the balanced subhypergraph decomposition problem in polynomial time. This follows immediately from the sign construction of a signed hypergraph from a given graph, and from Lemma 4.1 and Theorem 3.1.

Now we need only show that the bipartite subgraph decomposition problem and the graph 2^k-colorability problem are equivalent. In other words, a graph is 2^k-colorable, if and only if it can be *decomposed* into k bipartite subgraphs.

First, suppose that a graph G is decomposable into a set $\{G_i \mid i = 1, \ldots, k\}$ of bipartite subgraphs, and that (V_i^-, V_i^+) is a bipartition of G_i such that every edge connects a vertex in V_i^- to a vertex in V_i^+. We then assign to any vertex v a k-bit binary number with the ith bit determined as follows: It is 1 if v is in V_i^+, and it is 0 otherwise. Since any edge e is contained in at least one bipartite subgraph, the binary numbers assigned in this way to the two vertices incident to e differ in at least one bit. This leads to a

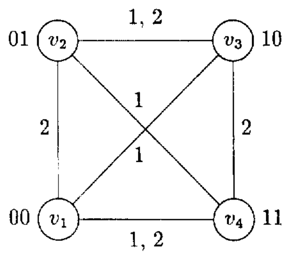


Fig. 8. A four-coloring of a graph.

valid 2^k-coloring. For example, in Fig. 8, graph G is decomposed into two bipartite subgraphs: G_1 induced by the edges labeled by 1, and G_2 induced by the edges labeled by 2. Subgraph G_1 is bipartite with respect to bipartition $(\{v_1, v_2\}, \{v_3, v_4\})$. Thus 0 is assigned to v_1 and v_2, and 1 to v_3 and v_4 for the first bit. Subgraph G_2 is bipartite with respect to bipartition $(\{v_1, v_3\}, \{v_2, v_4\})$. Thus 0 is assigned to v_1 and v_3, and 1 to v_2 and v_4 for the second bit. This results in the four-coloring of graph G, where v_1, v_2, v_3, and v_4 are assigned $00, 01, 10$, and 11, respectively. In summary, if G is decomposable into k bipartite subgraphs, then it is 2^k-colorable.

Conversely, suppose that a graph G is 2^k-colorable. The 2^k colors can be represented by k-bit binary numbers; then the two numbers assigned to two vertices incident to an edge differ in at least one bit. We say that the edge is *separated by that bit*. In Fig. 8, the bits that separate an edge are marked; for example, the edge that joins v_2 and v_3 is separated by bits 1 and 2. All the edges separated by the same bit induce a subgraph; thus there is a total of k subgraphs. According to the definition of coloring, each edge is separated by at least one bit, and hence contained in at least one subgraph. Therefore, the set of k subgraphs is a decomposition of the original graph. By construction, the bit value assigned to consecutive vertices in every cycle must alternate between 0 and 1. This means that every cycle in each subgraph is of even length; therefore each subgraph is bipartite.

Altogether, we have shown that a graph is 2^k-colorable, if and only if it can be decomposable into k bipartite subgraphs. Since the graph 2^k-colorability problem is NP-complete, so is the bipartite subgraph decomposition problem. Since the bipartite subgraph decomposition problem can be transformed into the balanced subhypergraph decomposition problem in polynomial time, the theorem is proved. □

6. Constrained via minimization

Constrained via minimization (CVM) is the original problem that motivated the introduction of the notion of signed hypergraph [23]. A complete derivation of the model of signed hypergraph from CVM and its related issues can be found in [19]. In this section, we provide a simple self-contained summary of the constrained via minimization problem and its signed hypergraph formulation. We note that the signed

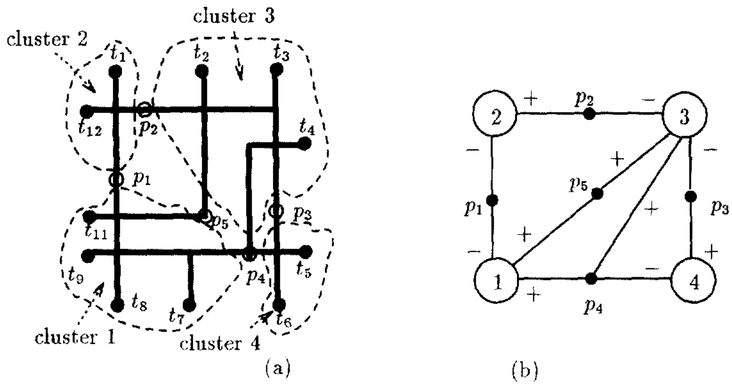


Fig. 9. (a) A routing (terminals: •, potential vias: ○); (b) Signed hypergraph.

hypergraph model was invented for the case of two routing layers; it is not directly applicable to the case in which multiple routing layers are available.

We describe the constrained via minimization problem by the example of Fig. 9(a). We are given a set P of *points* ($t_1,\ldots,t_{11}$ and $p_1,\ldots,p_5$ in the figure) in a plane, which are either *terminals* (t_i) or *potential vias* (p_i). We are also given a set N of (*wire*) *nets*, each net being a wire in the plane with multiple end-points and connecting several points from P. For example, $\{t_7,t_9,p_4\}$ represents a net. Two nets may *cross* each other (e.g., $\{t_7,t_9,p_4\}$ and $\{t_8,p_1\}$); this defines a *crossing relation* X on N. The *constrained via minimization* problem[2] is to assign nets to two layers in such a way that (1) no crossing nets appear in the same layer, and (2) the number of vias that connect nets assigned to different layers is minimized. As is usually done in practice, *we assume that there does exist a solution that satisfies condition* (1).

Clearly, two nets that cross each other should be assigned to different layers. Similarly, two nets that both cross a common third net should be assigned to the same layer. Such rules can be captured conveniently by the *cluster relation* C on N, which is defined to be the reflexive-and-transitive closure of the crossing relation X. Since the crossing relation is symmetric, the cluster relation is an equivalence relation on the set N of nets. Each equivalence class of C is called a *cluster* of nets. Once the layer assignment of any one net in a cluster is made, the layer assignment of all the other nets within the cluster is also determined. There are four clusters in Fig. 9(a).

By our assumption, in any cluster it is possible to assign nets to the two layers so that no two nets in the same layer cross. Therefore, for each cluster, we arbitrarily choose one set of nets as *reference* nets and assign to them the positive polarity $+$, and the other set as *non-reference* nets, with negative polarity $-$.

[2] The routing model implied by our mathematical framework is simple, concise and general. It contains the Manhattan routing, the knock-knee routing, and the more general gridless routing.

A cluster can have electrical connections to other clusters only through potential vias. (For example, cluster 1 is connected to clusters 2,3, and 4 through p_1, p_5, and p_4, respectively.) If we represent each cluster by a vertex, then each potential via corresponds to an edge. Since more than two clusters can meet at a potential via, an edge may connect more than two vertices, i.e., the natural concept required here is that of a hypergraph rather than just a graph. Furthermore, a potential via may be connected to a cluster through a positive or negative net; thus the hypergraph is naturally a signed hypergraph. For our example, Fig. 9(b) shows a signed hypergraph for the routing problem of Fig. 9(a).

We arbitrarily assign layer 1 to all the positive nets and layer 2 to all the negative nets. It is clear that vias p_2, p_3, and p_4 are required for this choice of reference nets. The condition for not requiring a real via at the position of a potential via is that all the nets connected to that potential via have the same polarity. To find an optimal layer assignment, we may have to interchange the reference nets with the non-reference nets in some of the clusters. For example, if we perform this interchange for clusters 2 and 4, only via p_1 is required in the resulting signed hypergraph.

In terms of the signed hypergraph of a routing, a solution assigning nets of the two layers corresponds to a partition of the vertices into two blocks. All vertices (clusters) for which the reference net assignment is the initial one are in one block of the partition, whereas those for which the reference net assignment is complemented are in the other block. If a bipartition balances an edge e, then all the vertices incident with e with one of the two polarities will have their reference net assignment interchanged. Consequently, no via will be required for that edge after the interchange. In other words, we want to partition all the vertices into two blocks so as to maximize the number of balanced edges.

By construction, the underlying graph for a given routing is planar. More precisely, it is a plane embedding of a planar graph [1]. On the other hand, given a planar signed hypergraph H, one can always construct a two-layer routing in such a way that the maximum balance problem of H can be solved by the constrained via minimization problem of that routing. Therefore, the constrained via minimization problem in a two-layer routing is equivalent to the maximum balance problem in a planar signed hypergraph.

We remark that the model of signed hypergraph gives for the first time a complete, yet simple, graph-theoretic model for constrained via minimization. Using this model, we have developed a linear-time heuristic that produced excellent experimental results on a set of benchmark examples [20].

7. Constrained logic encoding

Constrained encoding is a basic model for solving several problems in VLSI logic synthesis, such as race-free state assignment for asynchronous machines, delay-free state assignment for asynchronous machines without essential hazards, optimum state assignment for synchronous machines, and PLA decomposition [22]. In this section, we show that the constrained encoding problem can be cast as the minimum covering problem in a signed hypergraph.

Let $S=\{s_1,\dots,s_m\}$ be a set of states. A *(dichotomy) constraint* c on S is a pair (c^+,c^-) of disjoint subsets of S, where one subset may be empty. Let $B=\{0,1\}$. Given a set $S=\{s_1,\dots,s_m\}$ of m states and an integer k, a *binary encoding* $\boldsymbol{\alpha}$ of S is a mapping $\boldsymbol{\alpha}:S\to B^k$. We denote a component of $\boldsymbol{\alpha}$ by $\alpha:S\to B$, and call it a *bit assignment*.

A bit assignment $\alpha:S\to B$ is said to *satisfy* a constraint $c=(c^+,c^-)$ if and only if there exists $b\in B$ such that for all $s\in c^+$, $\alpha(s)=b$, and for all $s\in c^-$, $\alpha(s)=\bar{b}$, where $\bar{b}$ is the complement of b. An encoding $\boldsymbol{\alpha}:S\to B^k$ is said to *satisfy* a constraint c if and only if at least one bit assignment of $\boldsymbol{\alpha}$ satisfies c.

Given a set S of m states, and a set C of n constraints on S, the *constrained encoding problem* is to find a binary encoding $\boldsymbol{\alpha}$ of S that satisfies each constraint $c\in C$ and minimizes k. For example, consider constraints $(\{s_1,s_2,s_3\},\emptyset)$, $(\{s_1,s_3\},\{s_2\})$, $(\{s_1,s_3\},\{s_4\})$, $(\{s_3,s_4\},\{s_1\})$, and $(\{s_3,s_4\},\{s_2\})$. A minimum-length encoding satisfying all the constraints is $\boldsymbol{\alpha}=(100,101,110,011)$, where the bit vector assigned to s_1 is 100, etc. A two-bit encoding satisfying the largest number (4) of constraints is $\boldsymbol{\alpha}=(10,11,00,01)$.

To cast the constrained encoding problem in a graph-theoretic framework, we represent each state by a vertex and each constraint by an edge; then S and C form a signed hypergraph. For example, for $S=\{s_1,\dots,s_5\}$, the constraint $(\{s_1,s_2\},\{s_5\})$ corresponds to an edge with positive incidences with s_1 and s_2 and a negative incidence with s_5. A bit assignment is a bipartition of the set of vertices. A constraint is satisfied by a bit assignment if and only if the constraint's edge is balanced by the bit's bipartition. Therefore the constrained encoding problem is equivalent to the minimum covering problem of a signed hypergraph.

8. Related notions: Signed graphs and $(0,\pm 1)$-matrices

To our knowledge, the notion of signed hypergraph has not previously appeared in the literature, except in [19]. However, the equivalent concepts of $(0,\pm 1)$ matrices and of ± 1 weighted (also called directed) bipartite graphs are important concepts in the area of mathematical programming. In the graph-theoretical community, there have been studies on such topics as signed graphs [10]. In this section, we provide a brief summary of these concepts. Our emphasis is on their relations with signed hypergraphs.

8.1. Harary's signed graphs

In the study of certain phenomena in social psychology, Harary [10] conceived the notion of signed graph and it balance. A signed graph G consists of a set V of vertices together with two disjoint subsets E^+ and E^- of the set of unordered pairs of vertices. The elements of the sets E^+ and E^- are called positive edges and negative edges respectively. A cycle of a signed graph is positive if the number of negative edges involved is even; otherwise it is negative. A signed graph is balanced if all of its cycles are positive. Harary proved the following theorem: A signed graph G is balanced if and only if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 in such a way that each positive edge of G joins two vertices of the same subset and each negative edge joins two vertices of different subsets.

Clearly, our notions of signed hypergraph and the sign of a cycle, and our structure theorem generalize the ideas above. One slight difference is that we define the concept of balance in terms of bipartitions instead of cycles. This is because the VLSI synthesis problems that are of interest to us are naturally stated in terms of bipartitions. The absence of negative cycles is thus a structural property of these problems.

We make several remarks. First, our proof of the structure theorem is simple and provides a linear-time algorithm for balance testing. Second, as shown in [21], problems defined over signed graphs can be transformed exactly to problems in terms of weighted graphs. In this sense, signed graphs have the same "expressive" power as ordinary graphs. Signed hypergraphs, however, provide more expressive power than ordinary graphs.

8.2. *Restricted unimodularity and balanced* $(0, \pm 1)$ *matrices*

A matrix is totally unimodular (TU) if every square submatrix has determinant 0, or ± 1. It is well known that integer programs whose constraint matrices have the TU property can be solved optimally by relaxing the integrality restriction. This is important since integer programs are NP-complete, whereas their relaxations (linear programs) are polynomial-time solvable.

A very simple sufficient condition for total unimodularity is called restricted total unimodularity. A $(0, \pm 1)$ matrix ψ has restricted total unimodularity if and only if the corresponding ± 1 edge-weighted bipartite graph G of ψ is free of "negative" cycles. Here a "negative" cycle is one with the sum of the weights of all the involved edges congruent to 2 modulo 4, according to Conforti and Rao [7]. Alternatively, Yannakakis [28] defined the sign of a cycle to be the product of the signs of its edges, with the sign of an edge determined as follows: Suppose that G is bipartite with respect to bipartition (V^+, V^-), where V^- and V^- are two sets of vertices corresponding to all the edges and all the vertices, respectively. Then an edge is directed from a vertex v_i in V^+ to a vertex v_j in V^- if $\psi_{ij} = 1$, and to a vertex v_i in V^+ from a vertex v_j in V^- if $\psi_{ij} = -1$. Such a directed bipartite graph is called a *matrix digraph* by Yannakakis. Let C be a cycle; we traverse C in one direction, assign $+1$ to an edge e if e has the same direction, and assign -1 to e if e has the opposite direction.

Instead of associating signs with edges in ± 1 edge-weighted bipartite graph G as above, we can equivalently associate signs to "path segments" in the corresponding signed hypergraph as follows: The sign of a path segment $v_i e_j v_{i+1}$ is $-\psi(v_i, e_j)$

$\psi(v_{i+1}, e_j)$. This is similar to our definition in Section 2, whether the sign of a path segment $v_i e_j v_{i+1}$ is $+\psi(v_i, e_j)\psi(v_{i-1}, e_j)$. But this apparently slight difference leads to very distinct results: balance testing is much simpler than restricted-unimodularity testing. For balance testing, the absence of negative cycles can be checked by finding a bipartition, which is essentially one pass of breadth-first search. For unimodularity testing, the absence of "negative" cycles needs an examination of the entire cycle space – a much more involved process [7, 28].

Unimodularity testing for a signed graph G can be done by balance testing of its "negation" $\bar{G}$. The negation $\bar{G}$ of G is the same as G except that the sign of each edge is the negation of the original sign (here we use Harary's notion). It follows from Theorem 3.1 that, balance testing of $\bar{G}$ amounts to testing whether the set of vertices can be partitioned into two subsets so that any two vertices joined by a positive edge in $\bar{G}$ are in the same subset and any two vertices joined by a negative edge in $\bar{G}$ are in different subsets. In terms of G, this amounts to testing whether the set of vertices can be partitioned into two subsets so that any two vertices joined by a negative edge are in the same subset and any two vertices joined by a positive edge are in different subsets. In terms of the incidence matrix ψ of G, which has two nonzero entries in each column, ψ is totally unimodular if and only if the set of rows can be partitioned into two subsets, so that for every column with two 1's or two -1's, one nonzero entry is in each subset, and for every column with a 1 and a -1, both nonzero entries are in the same subset. This is a well-known result in mathematical programming [13].

Very recently, Conforti and Cornuéjols [6] reported that balanced $(0, \pm 1)$ matrices defined by Truemper [25] are a superclass of totally unimodular matrices. A *balanced* $(0, \pm 1)$ matrix is a $(0, \pm 1)$ matrix in which for every submatrix with exactly two nonzero entries per row and per column, the sum of the entries is a multiple of 4. In terms of the corresponding ± 1 edge-weighted bipartite graph, a $(0, \pm 1)$ matrix is balanced if and only if the sum of the weights of the edges in every chordless cycle is a multiple of 4. The problem of how to recognize such balanced $(0, \pm 1)$ matrices remains open.

9. Conclusions

A signed hypergraph is an extension of the conventional graph concept that (1) allows each edge to join any number of vertices, and (2) associates with each edge-vertex incidence either a positive sign or a negative sign. In this paper, we derived a characterization of the balance property of signed hypergraphs and established that a signed hypergraph is balanced if and only if it is free of negative cycles. Using this characterization, we studied two optimization problems, related to the balance property of signed hypergraphs, namely maximum balance and minimum covering. We established that both the maximum balance problem and the minimum covering problem are NP-complete. Moreover, we have shown that the maximum balance problem remains NP-complete even for planar signed hypergraphs. The two problems are natural models of two VLSI synthesis problems whose graph-theoretic formulations have not been explored in the past.

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