A CHARACTERIZATION OF FINITE TERNARY ALGEBRAS*

J. A. BRZOZOWSKI, J. J. LOU, R. NEGULESCU

Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
E-mail: {brzozo,jjlou,radu}@maveric.uwaterloo.ca

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A ternary algebra is a De Morgan algebra (that is, a distributive lattice with 0 and 1 and a complement operation that satisfies De Morgan's laws) with an additional constant Φ satisfying $\phi = \overline{\phi}$, $(a + \overline{a}) + \phi = a + \overline{a}$, and $(a * \overline{a}) * \phi = a * \overline{a}$. We provide a characterization of finite ternary algebras in terms of "subset-pair algebras," whose elements are pairs (X, Y) of subsets of a given base set $\mathcal{E}$, which have the property $X \cup Y = \mathcal{E}$, and whose operations are based on common set operations.

1. Introduction

In 1959, Muller defined a ternary algebra on the set $\{0, 1/2, 1\}$ [6] for studying transient phenomena in switching circuits. He noted that this algebra is equivalent to the strong 3-valued logic of Kleene, and satisfies Kleene's laws $(a * \overline{a}) + b + \overline{b} = b + \overline{b}$ and $a * \overline{a} * (b + \overline{b}) = a * \overline{a}$. In terms of lattice theory, this system is a De Morgan algebra [1] with the additional constant 1/2, which is denoted by Φ in this paper. In 1964, Yoeli and Rinon applied ternary algebra to the study of static hazards in combinational switching circuits [9]. In 1965, Eichelberger used ternary algebra for analyzing various hazard phenomena in combinational and sequential switching circuits [3]. In 1983, Mukaidono studied and characterized certain ternary functions (also called regular functions), and gave a set of laws for ternary algebra [5]. He called the resulting algebra Kleene algebra with a center (denoted here by Φ). In 1995, Brzozowski and Seger defined a ternary algebra to be a De Morgan algebra with an additional constant Φ satisfying $\Phi = \overline{\Phi}$, $(a + \overline{a}) + \Phi = a + \overline{a}$, and $(a * \overline{a}) * \Phi = a * \overline{a}$ [2]. They pointed out that the last two laws are equivalent to Kleene's laws; they also described additional applications of ternary

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algebra. Also in 1995, Negulescu proposed process spaces, a formalism for modeling interacting systems [7]. In this paper, we generalize process spaces to subset-pair algebras, and give a characterization of finite ternary algebras using subset-pair algebras.

In Sec. 2, we state a set of laws for ternary algebras. Also, to provide examples of ternary algebras, we define process spaces and some of their operations [7], and we prove that the resulting algebraic systems are ternary algebras. In Sec. 3, we summarize some results from lattice theory; these results will be needed in the sequel. In Sec. 4 we link ternary algebra to lattice theory and give some properties of ternary algebras. In Sec. 5 we define subset-pair algebras and prove that every ternary algebra is isomorphic to a subset-pair algebra, and vice versa. Section 6 concludes the paper.

2. Ternary Algebras

A *ternary algebra* is an algebraic system $T = \langle \mathcal{A}, +, *, ^{-}, 0, \Phi, 1 \rangle$, where $\mathcal{A}$ is a set, $+$ and $*$ are binary operations on $\mathcal{A}$, $^{-}$ is a unary operation on $\mathcal{A}$, and 0, Φ and 1 are constants in $\mathcal{A}$, such that the following laws are satisfied [2]:

For all a, b and c in $\mathcal{A}$, we have:

T1 $a + a = a$	T1′ $a * a = a$
T2 $a + b = b + a$	T2′ $a * b = b * a$
T3 $a + (b + c) = (a + b) + c$	T3′ $a * (b * c) = (a * b) * c$
T4 $a + (a * b) = a$	T4′ $a * (a + b) = a$
T5 $a + 0 = a$	T5′ $a * 1 = a$
T6 $a + 1 = 1$	T6′ $a * 0 = 0$
T7 $\overline{(\overline{a})} = a$	
T8 $a + (b * c) = (a + b) * (a + c)$	T8′ $a * (b + c) = (a * b) + (a * c)$
T9 $\overline{(a + b)} = \overline{a} * \overline{b}$	T9′ $\overline{(a * b)} = \overline{a} + \overline{b}$
T10 $(a + \overline{a}) + \Phi = a + \overline{a}$	T10′ $(a * \overline{a}) * \Phi = a * \overline{a}$
T11 $\overline{\Phi} = \Phi$	

These laws are not independent; they are modeled after a similar set of laws for Boolean algebras. Throughout this paper, we will be concerned only with finite ternary algebras. Now we give an example of a ternary algebra.

Example. Let $\mathcal{E}$ be an arbitrary set. Let $\mathcal{S}_{\mathcal{E}}$ be the set of all ordered pairs (X, Y) of subsets of $\mathcal{E}$ such that $X \cup Y = \mathcal{E}$. The set $\mathcal{S}_{\mathcal{E}}$ is called the *process space* over $\mathcal{E}$ [7]. Some of the operations on $\mathcal{S}_{\mathcal{E}}$ are:

- *join*, denoted by $\sqcup$:

$$(X_1, Y_1) \sqcup (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cup Y_2);$$

- *meet*, denoted by $\sqcap$:

$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cup X_2, Y_1 \cap Y_2);$$

- *reflection*, denoted by $-$:

$$-(X,Y) = (Y,X)\,.$$

We denote $(\emptyset,\mathcal{E})$ by $\top$, $(\mathcal{E},\emptyset)$ by $\perp$ and $(\mathcal{E},\mathcal{E})$ by Φ.

Proposition 1. *Let $\mathcal{S}_\mathcal{E}$ be a process space. Then $\langle \mathcal{S}_\mathcal{E}, \sqcup, \sqcap, -, \perp, \Phi, \top\rangle$ is a ternary algebra.*

Proof. We need to check all the laws of ternary algebras. Most of these proofs are straightforward. We show the proofs of T8 and T10 as examples. Let

$$p = (X_1, Y_1)\,, \quad q = (X_2, Y_2)\,, \text{ and } r = (X_3, Y_3)\,.$$

T8:

$$\begin{aligned}
p \sqcup (q \sqcap r) &= (X_1, Y_1) \sqcup ((X_2, Y_2) \sqcap (X_3, Y_3))\\
&= (X_1, Y_1) \sqcup (X_2 \cup X_3, Y_2 \cap Y_3)\\
&= (X_1 \cap (X_2 \cup X_3), Y_1 \cup (Y_2 \cap Y_3))\\
&= ((X_1 \cap X_2) \cup (X_1 \cap X_3), (Y_1 \cup Y_2) \cap (Y_1 \cup Y_3))\\
&= (X_1 \cap X_2, Y_1 \cup Y_2) \sqcap (X_1 \cap X_3, Y_1 \cup Y_3)\\
&= (p \sqcup q) \sqcap (p \sqcup r)\,;
\end{aligned}$$

T10:

$$\begin{aligned}
(p \sqcup -p) \sqcup \Phi &= ((X_1, Y_1) \sqcup (Y_1, X_1)) \sqcup (\mathcal{E}, \mathcal{E})\\
&= (X_1 \cap Y_1, Y_1 \cup X_1) \sqcup (\mathcal{E}, \mathcal{E})\\
&= (X_1 \cap Y_1 \cap \mathcal{E}, Y_1 \cup X_1 \cup \mathcal{E})\\
&= (X_1 \cap Y_1, \mathcal{E})\\
&= (X_1 \cap Y_1, Y_1 \cup X_1)\\
&= (X_1, Y_1) \sqcup (Y_1, X_1)\\
&= p \sqcup -p\,.
\end{aligned}$$

□

3. Fundamentals of Lattice Theory

We now give a brief overview of lattice theory, following [8]. A *partially ordered set* (*poset*) is an algebraic system $\langle \mathcal{P}, \geq \rangle$ where $\mathcal{P}$ is a set and $\geq$ is a binary relation on $\mathcal{P}$, which satisfies the following properties:

P1. $x \geq x$ for all x in $\mathcal{P}$ (reflexive property);
P2. $x \geq y \wedge y \geq x \Rightarrow x = y$ for all x, y in $\mathcal{P}$ (antisymmetric property);
P3. $x \geq y \wedge y \geq z \Rightarrow x \geq z$ for all x, y, z in $\mathcal{P}$ (transitive property).

We say $x \leq y$ if $y \geq x$, and $x > y$ if $x \geq y$ and $x \neq y$. For any two elements x and y in a lattice, a finite chain $x = a_1 > a_2 > \cdots > a_n = y$ is called a *maximal chain* from x to y if, for $i = 1, 2, \ldots, n-1$, there does not exist b_i such that $a_i > b_i > a_{i+1}$.

We can use a *Hasse diagram* to represent a poset $\mathcal{P}$. In such a diagram, the elements of $\mathcal{P}$ are points and the relation $x \geq y$ is depicted by a descending line from x to y. An element x of $\mathcal{P}$ is called an *upper bound* of a subset S of $\mathcal{P}$ if $x \geq y$ for all $y \in S$. An upper bound x of S is called the *least upper bound* (lub) if, for every upper bound x' of S, we have $x' \geq x$. By replacing $\geq$ with $\leq$ in the definition above, we can define *lower bound* and *greatest lower bound* (glb).

A *lattice* is a poset in which every two elements have a lub and a glb. We denote the lub of x and y by $x + y$ and the glb of x and y by $x * y$. The following is an equivalent definition of a lattice. Any set $\mathcal{L}$ with two binary operations, denoted by $+$ and $*$, satisfying laws T2, T2′, T3, T3′, T4, T4′ is a *lattice*. Note that law T1 is implied by these laws because, for any y, $x = x + (x * (x + y)) = x + x$. Similarly law T1′ is also implied by these laws. From this second definition, we see that ternary algebras are lattices and $a \geq b$ means $a + b = a$. In fact, ternary algebras are distributive lattices because of the distributive laws T8 and T8′.

A lattice is said to be *complete* if every subset has a lub and a glb. We will be concerned only with finite lattices, and finite lattices are all complete. A complete lattice has a *null* element 0, which is the glb of all the elements in the lattice, and a *universal* element 1, which is the lub of all the elements in the lattice.

A *sublattice* of a lattice $\mathcal{L}$ is a subset $\mathcal{M}$ of $\mathcal{L}$ such that

$$x \in \mathcal{M} \wedge y \in \mathcal{M} \Rightarrow x + y \in \mathcal{M} \wedge x * y \in \mathcal{M}.$$

Distributive lattices satisfy the following two theorems:

Theorem 1. *A lattice $\mathcal{L}$ is distributive if and only if neither of the lattices in Fig. 1 (a) and (b) is a sublattice of $\mathcal{L}$.*

(a) (b)

Fig. 1. Nondistributive lattices of five elements.

Theorem 2. (Jordan-Hölder) *Let $x = a_1 > a_2 > \cdots > a_n = y$ and $x = b_1 > b_2 > \cdots > b_m = y$ be two maximal chains from x to y of a distributive lattice; then $n = m$.*

4. Cardinality of Ternary Algebras

In this section, we give some properties of ternary algebras. As we mentioned in the previous section, ternary algebras are lattices with partial order $\geq$ defined by $a \geq b$ if $a + b = a$.

Proposition 2. *Let a and b be two elements in a ternary algebra. Then*
(a) $a \geq b$ *if and only if* $\overline{b} \geq \overline{a}$;
(b) $a \geq \Phi$ *if and only if* $\Phi \geq \overline{a}$.

Proof. (a) Suppose $a \geq b$. By definition, this means $a + b = a$. Taking the complement of each side, by law T9 we get $\overline{a} * \overline{b} = \overline{a}$. Thus, $\overline{a} + \overline{b} = \overline{a} * \overline{b} + \overline{b} = \overline{b}$, that is, $\overline{b} \geq \overline{a}$. Similarly, we can prove $a \geq b$ if $\overline{b} \geq \overline{a}$. This proves (a).

(b) Now suppose $a \geq \Phi$. By (a), we get $\overline{\Phi} \geq \overline{a}$, and by T11, $\Phi \geq \overline{a}$. On the other hand, if $\Phi \geq \overline{a}$, again by (a) and T11, we get $a \geq \Phi$. This proves (b). □

Proposition 3. *Every finite ternary algebra has an odd number of elements.*

Proof. First we prove that $a = \overline{a}$ implies $a = \Phi$ in a ternary algebra. From T10,

$$a + \Phi = a + \overline{a} + \Phi = a + \overline{a} = a\,.$$

This means $a \geq \Phi$. Since $a = \overline{a}$, we also have $\overline{a} \geq \Phi$. By Proposition 2, we have $\Phi \geq a$. Therefore, by the antisymmetric property of partial ordering, we get $a = \Phi$.

Now we can express the set of elements of a ternary algebra as the union of sets of the form $\{a, \overline{a}\}$. Exactly one of these sets has one element; that happens when $a = \Phi$. Therefore every finite ternary algebra has an odd number of elements. □

Proposition 4. *For every odd number $n \geq 3$, there is a ternary algebra with n elements. In fact, for every odd number $n \geq 3$,*

$$A_n = \langle \{0, 1, \dots, n-1\}, \max, \min, ^{-}, 0, (n-1)/2, n-1 \rangle\,,$$

where $\bar{a} = n - 1 - a$, is a ternary algebra.

Proof. Omitted, as it is straightforward. □

The smallest ternary algebra is $\langle \{0, \Phi, 1\}, +, *, ^{-}, 0, \Phi, 1 \rangle$; it has three elements and is isomorphic to A_3 as defined above. Using the propositions above and the two theorems in the previous section, one can verify that, up to isomorphism, there is exactly one ternary algebra with five elements, there are two with seven elements, and four with nine elements.

5. Subset-Pair Algebras

Let $\mathcal{E}$ be a finite non-empty set, and let $\mathcal{P}$ be a collection of ordered pairs (X, Y) of subsets of $\mathcal{E}$ such that $X \cup Y = \mathcal{E}$. We call $\mathcal{P}$ a *subset-pair algebra* if it is closed under $\sqcup$, $\sqcap$ and $-$ as defined in Sec. 1, and if $\top = (\emptyset, \mathcal{E})$, $\Phi = (\mathcal{E}, \mathcal{E})$, and $\bot = (\mathcal{E}, \emptyset)$

are in $\mathcal{P}$. Subset-pair algebras generalize process spaces, and every subset-pair algebra is a subalgebra of a process space.

As we have seen, process spaces are ternary algebras. Since $\mathcal{P}$ is a subalgebra of a process space, the special elements $\top$, Φ and $\perp$ are in $\mathcal{P}$ and all the laws for ternary algebras hold for $\mathcal{P}$. Thus we have the following corollary to Proposition 1:

Corollary. *Every subset-pair algebra is a ternary algebra.*

The converse of the corollary above is also true for finite ternary algebras. The proof is by construction of a subset-pair algebra isomorphic to the given ternary algebra; the construction uses subdirect product decompositions [8].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two lattices. The set of all ordered pairs (x, y) with $x \in \mathcal{X}$, $y \in \mathcal{Y}$ forms a lattice if we define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\,,$$

$$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)\,.$$

Laws T2, T3, T4, T2′, T3′ and T4′ can be easily verified, as both $+$ and $*$ operations are done component-wise. This lattice we constructed is called the *direct product* of $\mathcal{X}$ and $\mathcal{Y}$, and is denoted by $\mathcal{X} \times \mathcal{Y}$. The partial order for this lattice is

$$(x_1, y_1) \geq (x_2, y_2) \Leftrightarrow x_1 \geq x_2 \wedge y_1 \geq y_2\,.$$

Let $\mathcal{L}$ be a finite distributive lattice with at least three elements. Let a be an element in $\mathcal{L}$ distinct from 0 and 1. Then the elements of the form $a + x$ form a sublattice of $\mathcal{L}$, call it $\mathcal{L}_1(a)$, or simply $\mathcal{L}_1$, if there is no danger of ambiguity. Note that the null element of $\mathcal{L}_1$ is a. Similarly the elements of the form $a * x$ form a sublattice of $\mathcal{L}$, call it $\mathcal{L}_0(a) = \mathcal{L}_0$. Note that the universal element of $\mathcal{L}_0$ is a. The set $\mathcal{K}$ of ordered pairs $(a + x, a * x)$ is a subset of the direct product $\mathcal{L}_1 \times \mathcal{L}_0$. Further, if $(a + x, a * x) = (a + y, a * y)$, then

$$\begin{aligned} x &= x + (x * a) \\ &= x + (y * a) \\ &= (x + y) * (x + a) \\ &= (x + y) * (y + a) \\ &= y + x * a \\ &= y + y * a \\ &= y\,. \end{aligned}$$

Therefore, there is a bijection between set $\mathcal{K}$ and the elements of lattice $\mathcal{L}$. Moreover

$$\begin{aligned} (a + x, a * x) + (a + y, a * y) &= (a + x + a + y, (a * x) + (a * y)) \\ &= (a + (x + y), a * (x + y))\,, \end{aligned}$$

and similarly

$$(a+x, a*x)*(a+y, a*y) = (a+(x*y), a*(x*y)).$$

Thus $\mathcal{K}$ is a sublattice of the direct product $\mathcal{L}_0 \times \mathcal{L}_1$ and $\varphi : x \leftrightarrow (a+x, a*x)$ determines an isomorphism between $\mathcal{L}$ and this sublattice. We say that $\mathcal{L}$ is isomorphic to a subdirect product of $\mathcal{L}_0$ and $\mathcal{L}_1$. Suppose that the maximal chains from the universal to the null elements in $\mathcal{L}$, $\mathcal{L}_0$, and $\mathcal{L}_1$ have lengths n, n_0, and n_1, respectively. Then n_0 is the length of a maximal chain from a to 0, and n_1 is the length of a maximal chain from 1 to a. Thus, by the Jordan–Hölder Theorem, we must have $n = n_0 + n_1$. If $\mathcal{L}_0$ or $\mathcal{L}_1$ has more than two elements, the procedure above can be repeated and we can express $\mathcal{L}_0$ or $\mathcal{L}_1$ as the subdirect product of two lattices, both of which have a smaller maximal chain length from their universal to their null elements than $\mathcal{L}_0$ or $\mathcal{L}_1$. We can repeat this process as long as one of the lattices in the direct product has more than two elements. This decomposition terminates since n is finite. In the end, we can write $\mathcal{L}$ as a subdirect product of n 2-element lattices. Each of these 2-element lattices is isomorphic to the lattice consisting only of 0 and 1, and each element of the subdirect product may be written as an ordered n-tuple of 0's and 1's.

Theorem 3. *Every finite ternary algebra is isomorphic to a subset-pair algebra.*

Proof. Given a finite ternary algebra, we prove the claim by constructing a subset-pair algebra that is isomorphic to the ternary algebra. Let $\mathcal{L}$ be the lattice representing the ternary algebra. We define $\mathcal{L}_1$ and $\mathcal{L}_0$ as before with $a = \Phi$. From Proposition 2, we see that $x \leftrightarrow \overline{x}$ defines an isomorphism from $\mathcal{L}_1$ to $\mathcal{L}_0$. Now $\mathcal{L}_0$ is a subdirect product of lattices consisting of 0 and 1. Assuming $1 \geq 0$, we can label each element in the sublattice $\mathcal{L}_0$ using an ordered n-tuple of 0's and 1's, where n is the length of any maximal chain from Φ to 0. Further, Φ is labeled with the n-tuple consisting of 1's, 0 is labeled with the n-tuple consisting of 0's. We shall simply consider the elements of $\mathcal{L}_0$ to be these n-tuples. Here, the partial order is determined by component-wise comparison, i.e. if $x = (j_1, j_2, \ldots, j_n)$ and $y = (k_1, k_2, \ldots, k_n)$, then $x \geq y$ means $j_i \geq k_i$ for $i = 1, 2, \ldots, n$. Also, $x + y = (l_1, l_2, \ldots, l_n)$ where $l_i = \max(j_i, k_i)$, and $x * y = (m_1, m_2, \ldots, m_n)$ where $m_i = \min(j_i, k_i)$. We use the isomorphism

$$\psi : (j_1, j_2, \ldots, j_n) \leftrightarrow \{a_i \mid j_i = 1\}.$$

to convert each element of $\mathcal{L}_0$ into a subset of the set $\mathcal{E} = \{a_1, a_2, \ldots, a_n\}$. Now we observe that $\psi(l_1 + l_2) = \psi(l_1) \cup \psi(l_2)$ and $\psi(l_1 * l_2) = \psi(l_1) \cap \psi(l_2)$, where l_1 and l_2 are elements of $\mathcal{L}_0$. We can assign n-tuples to the elements in the sublattice $\mathcal{L}_1$ in such a way that each element x is assigned the same n-tuple as $\overline{x}$ in the sublattice $\mathcal{L}_0$. Here, we assume $0 \geq 1$ and again partial order is determined by component-wise comparison. We use the isomorphism ψ to convert elements of $\mathcal{L}_1$ into subsets of $\mathcal{E}$ and observe that $\psi(l_1 + l_2) = \psi(l_1) \cap \psi(l_2)$ and $\psi(l_1 * l_2) = \psi(l_1) \cup \psi(l_2)$, where l_1 and l_2 are elements of $\mathcal{L}_1$.

Now take the direct product $\mathcal{L}_1 \times \mathcal{L}_0$. We know that $\mathcal{L}$ is isomorphic to a sublattice of this direct product and the isomorphism is $\varphi : x \leftrightarrow (\Phi + x, \Phi * x)$. We define the following function from $\mathcal{L}$ to $\{(X,Y) \mid X, Y \subseteq \mathcal{E}\}$:

$$\vartheta : x \rightarrow (\psi(\Phi + x), \psi(\Phi * x)) .$$

Let $\mathcal{L}'$ be the image of $\mathcal{L}$ under ϑ. We shall prove that $\mathcal{L}'$ is a subset-pair algebra, and ϑ is an isomorphism. First we prove that every pair (X, Y) of subsets in $\mathcal{L}'$ satisfies $X \cup Y = \mathcal{E}$. Suppose $(X,Y) = \vartheta(a)$. Then $(X,Y) = (\psi(\Phi + a), \psi(\Phi * a))$. From our construction, we know that $\psi(\Phi + a) = \psi(\Phi * \overline{a})$, since $\overline{(\Phi + a)} = \Phi * \overline{a}$. Note that both $\Phi * \overline{a}$ and $\Phi * a$ are elements of the sublattice $\mathcal{L}_0$. Thus we have

$$\begin{aligned} X \cup Y &= \psi(\Phi + a) \cup \psi(\Phi * a) \\ &= \psi(\Phi * \overline{a}) \cup \psi(\Phi * a) \\ &= \psi((\Phi * \overline{a}) + (\Phi * a)) \\ &= \psi(\Phi * (\overline{a} + a)) \\ &= \psi(\Phi * (\overline{a} + a + \Phi)) \\ &= \psi(\Phi * (\overline{a} + a) + \Phi) \\ &= \psi(\Phi) \\ &= \mathcal{E} . \end{aligned}$$

Therefore, we have $X \cup Y = \mathcal{E}$ as required. Now we show that ϑ is an isomorphism. Since φ is an isomorphism and ψ is injective on $\mathcal{L}_0$, it follows that ϑ is injective. From the definition of $\mathcal{L}'$, we see that ϑ is surjective. Thus ϑ is a bijection. Let a and b be two elements of the ternary algebra. We have $\Phi + a, \Phi + b \in \mathcal{L}_1$; $\Phi * a, \Phi * b \in \mathcal{L}_0$. Thus

$$\begin{aligned} \vartheta(a) \sqcup \vartheta(b) &= (\psi(\Phi + a), \psi(\Phi * a)) \sqcup (\psi(\Phi + b), \psi(\Phi * b)) \\ &= (\psi(\Phi + a) \cap \psi(\Phi + b), \psi(\Phi * a) \cup \psi(\Phi * b)) \\ &= (\psi(\Phi + a + \Phi + b), \psi((\Phi * a) + (\Phi * b))) \\ &= (\psi(\Phi + (a + b)), \psi(\Phi * (a + b))) \\ &= \vartheta(a + b) . \end{aligned}$$

Similarly we can prove $\vartheta(a) \sqcap \vartheta(b) = \vartheta(a * b)$. Also, we have

$$\begin{aligned} -\vartheta(a) &= -(\psi(\Phi + a), \psi(\Phi * a)) \\ &= (\psi(\Phi * a), \psi(\Phi + a)) \\ &= (\psi(\overline{\Phi + \overline{a}}), \psi(\overline{\Phi * \overline{a}})) \\ &= (\psi(\Phi + \overline{a}), \psi(\Phi * \overline{a})) \\ &= \vartheta(\overline{a}) . \end{aligned}$$

Thus ϑ is a homomorphism. Since ϑ is bijective, it is an isomorphism. We also see that $\mathcal{L}'$ is closed under $\sqcup$, $\sqcap$, and $-$. Finally, we have $(\emptyset, \mathcal{E}) = \vartheta(1)$, $(\mathcal{E}, \mathcal{E}) = \vartheta(\Phi)$, and $(\mathcal{E}, \emptyset) = \vartheta(0)$. Therefore $\mathcal{L}'$ is a subset-pair algebra. This proves our theorem.

6. Concluding Remarks

In this paper, we studied ternary algebras, and we defined subset-pair algebras. Our main result shows that every finite ternary algebra is isomorphic to a subset-pair algebra. This provides a complete characterization of finite ternary algebras. With this result, properties of ternary algebras can be checked using set-theoretic means, since the operations of subset-pair algebras are based on common set operations.

Recently, our characterization theorem has been generalized to infinite ternary algebras by Z. Ésik, who also gave two Cayley representations for ternary algebras [4].

Subset-pair algebras are subalgebras of process spaces. Since process spaces are distributive lattices, each subset-pair algebra is a sublattice of a process space. Consequently, every finite ternary algebra is a sublattice of a process space.

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