GRAPH CONGRUENCES AND PAIR TESTING (*)

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Abstract. — This paper considers the congruence \( \sim \) on a free monoid where \( u \sim v \) iff \( u \) and \( v \) have the same letters and the same ordered pairs of letters. The motivation for this comes from the study of bi-locally testable languages defined by testing pairs of words. As in the case of locally testable languages, a theorem on graph congruences is used in order to obtain a characterization of the family of bi-locally testable languages. Such a theorem on graph congruences is developed in this paper.

Résumé. — Dans cet article on considère la congruence sur un monoïde libre telle que deux mots soient équivalents si ils contiennent les mêmes lettres et les mêmes couples ordonnés de lettres. L'étude de cette congruence est motivée par l'étude des langages bi-localement testables. Comme dans le cas des langages localement testables, on démontre un théorème sur les congruences de graphes pour caractériser la classe des langages bi-localement testables.

1. INTRODUCTION

The family of locally testable languages plays a key role in the study of star-free languages. It is defined as follows: The membership of a word \( w \) in a language \( L \) is uniquely determined by the prefix of length \( k-1 \) of \( w \), the suffix of length \( k-1 \) of \( w \), and the set of all segments of length \( k \) appearing in \( w \), where \( k \geq 1 \) is an integer depending on \( L \). The syntactic semigroup \( S \) that corresponds to a locally testable language \( L \) satisfies the condition that for each idempotent \( e \in S \), the monoid \( eS e \) is idempotent and commutative. Conversely if \( S \) is the syntactic semigroup of \( L \) and \( S \) is finite and satisfies the above-mentioned conditions on \( eS e \), then \( L \) is locally testable. The proof of this last statement is quite difficult. One of the key steps in this proof is a theorem on graphs. This theorem, due to Simon, appeared originally in [2], though it was not formulated as a separate result on graphs. The treatment

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of the theorem as a theorem on directed graphs is due to Eilenberg [3]. The theorem involves a congruence $\sim_k$ that corresponds to $k=1$ in the theorems described above. More precisely, the prefix and suffix are not tested (since $k-1=0$), and only segments of length one (i.e., letters) are considered.

The next family in the hierarchy of languages of depth one [1], after the locally testable family, is that of bi-locally testable languages. Membership of a word $w$ in a bi-locally testable language is determined by the prefix and suffix of length $k-1$ of $w$, and by the set of ordered pairs of segments of length $k$ that appear in $w$. The characterization of syntactic semigroups of bi-locally testable languages is due to Knast [4], and uses the theorem on graphs presented in this paper as one of the basic steps. The theorem involves the congruence $\sim_k$ that again corresponds to $k=1$. This time, however ordered pairs of letters are used.

2. THE MAIN THEOREM

We first briefly recall Eilenberg's notation for graphs [3].

A directed graph $G$ consists of two possibly infinite sets $V$ (vertices) and $E$ (edges) along with two functions:

$$\alpha, \omega : E \rightarrow V.$$ 

If $e$ is an edge, $e\alpha$ and $e\omega$ are the initial and final vertices of $e$. Two edges $e_1$ and $e_2$ are consecutive iff $e_2 \alpha = e_1 \omega$. Let $E^*(E^*)$ be the free semigroup (free monoid) generated by $E$, and let $C \subseteq E^2$ be the set of words $e_1, e_2$ such that $e_1$ and $e_2$ are non-consecutive. The set of (non-empty) paths of $G$ is then:

$$P = E^* - E^*CE^*.$$ 

If $p = e_1 \ldots e_n$ is a path, define $p\alpha = e_1\alpha$ and $p\omega = e_n\omega$. The length of the path is $|p| = n$, where $n \geq 1$. A path $p$ is a loop about vertex $v$ iff $v = p\alpha = p\alpha$.

If $p = e_1 \ldots e_n$, $q = e'_1 \ldots e'_m$, and $p\omega = q\alpha$ then $p$ and $q$ are consecutive and $pq = e_1 \ldots e_ne'_1 \ldots e'_m$ is a path. For any vertex $v$, $l_v$ is a loop of length about $v$, i.e. $l_v\alpha = l_v\omega = v$. For technical reasons we assume that the set $\{ l_v \mid v \in V \}$ of trivial paths is adjoined to $P$. Two paths $p$ and $p'$ are coterminous iff $p\alpha = p'\alpha$ and $p\omega = p'\omega$. An equivalence relation $\sim$ on $P$ is a congruence iff:

(i) $p \sim p'$ implies $p$ and $p'$ are coterminous.

(ii) If $p \sim p'$, $q \sim q'$ and $p$ and $q$ are consecutive, then $pq \sim p'q'$.
Let $\tau: E^* \rightarrow 2^E$ be the function that associates with each word $w$ in $E^*$ the set of edges (letters) appearing in $w$:

$$w \tau = \{ e \in E \mid w \equiv w_1 e w_2 \text{ for some } w_1, w_2 \in E^* \}.$$ 

Similarly let $w \tau_2$ be the set of ordered pairs of edges in $w$:

$$w \tau_2 = \{ (e_1, e_2) \in E \times E \mid w \equiv w_0 e_1 w_1 e_2 w_2, w_0, w_1, w_2 \in E^* \}.$$ 

We define the following congruence on $E^*$. Given $x, y \in E^*$:

$$x_2 \sim y \iff x \tau_2 = y \tau_2 \text{ and } x \tau = y \tau.$$ 

If $p$ is a path of length $> 0$, then $p \tau$ and $p \tau_2$ are defined as above. If $p = 1_v$ for some $v \in V$ then $p \tau = p \tau_2 = \emptyset$.

**Theorem** Let $\sim$ be the smallest congruence on $P$ satisfying:

$$z_1 (pq)^2 p z_2 (sr)^2 s z_2 \sim z_1 (pq)^2 z' (sr)^2 s z_2,$$  

(1)

for all $p, q, r, s, z_1, z_2, z, z' \in P$ such that:

$$z \tau \subseteq z_1 \tau \cap z_2 \tau \text{ and } z' \tau \subseteq z_1 \tau \cap z_2 \tau,$$

then for any two coterminal paths $x$ and $y$ the conditions $x \sim y$ and $x_2 \sim y$ are equivalent.

The proof of this result is the subject of the rest of this paper. Before proceeding with the proof we make the following comments. The congruence $\sim$ involves testing the set $w \tau_2$ of pairs of letters appearing in a word $w$ (or the set $w \tau$ in case $w \tau_2 = \emptyset$, i.e. $|w| \leq 1$), and is defined on $E^*$. The theorem states that the equivalence of any two coterminal paths with respect to $\sim$ can always be demonstrated by coterminal path transformations of the form (1). It is easily verified that:

$$x \sim y \text{ implies } x_2 \sim y.$$  

(2)

The converse of (2) constitutes the problem.

Rule (1) is quite complex as compared to the rules in Simon's theorem, where the rules corresponding to (1) are:

$$x \sim x^2 \quad \text{and} \quad xy \sim yx,$$

for any two coterminal loops $x$ and $y$. We were unable to simplify Rule (1) or to replace it by a set of equivalent or weaker rules. The graph of Figure 1

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provides an example of the difficulty involved. Consider the coterminated paths
\[ x = c' \, d_1 \, c d_2 \, (a_1 \, a_2)^2 \, a_1 \, c b_1 \, (b_2 \, b_1)^2 \, e_1 \, c e_2 \, c' \]
and
\[ y = c' \, d_1 \, c d_2 \, (a_1 \, a_2)^2 \, c' \, (b_2 \, b_1)^3 \, e_1 \, c e_2 \, c'. \]
One easily verifies that \( x \sim y \). If we let \( z_1 = c' \, d_1 \, c d_2 \) and \( z_2 = e_1 \, c e_2 \, c' \), we have an instance where Rule (1) applies. We were unable to find a simpler set of rules for this example.

![Figure 1](image)

In a number of cases Rule (1) degenerates to considerably simpler rules. It will be convenient to identify them distinctly, even though they are covered by (1). If \( z, z' \subseteq z_1 \cap z_2 \) then:
\[
\begin{align*}
  z_1 z z_2 & \sim z_1 z' z_2, \quad (1\,a')  \\
  z_1 (pq)^2 pzz_2 & \sim z_1 (pq)^3 z' z_2, \quad (1\,b')  \\
  z_1 z r (sr)^2 z_2 & \sim z_1 z' (sr)^2 z_2. \quad (1\,c')
\end{align*}
\]

3. SINGULARITIES

Let \( A \) be a finite alphabet and \( x \in A^* \). If \( x = x_1 a x_2 \), \( a \in A \) and \( a \notin (x_1, x_2) \) then \( a \) is a singular letter of \( x \). If \( x = x_0 a x_1 b x_2 \) where \( a \) and \( b \) are not singular letters of \( x \) and \( (b, a) \notin x_0 x_2 \), then \((a, b)\) is a singular pair of \( x \). Singular letters and singular pairs are called singularities of \( x \). If \( x = x_0 a x_1 b x_2 \), this factorization is an occurrence of \((a, b)\). An occurrence is inner if \( a \notin x_1, b \notin x_1 \). Clearly every singular pair \((a, b)\) has a unique inner occurrence.
consisting of the rightmost $a$ of $x$ and the leftmost $b$. An occurrence $x_0 ax_1 bx_2$ is proper if $ax_1$ and $x_1 b$ have no singularities of $x$; note that every proper occurrence is necessarily inner. A singular pair need not necessarily have a proper occurrence. For example, let $x = aeabacdâ'fâ's$. Then $e$ is the only singular letter of $x$ and $(a, c), (a, d), (a, f), (b, c), (b, d), (b, f)$ are the singular pairs of $x$. The factorization $(aeb)b(ae)d(fdâ's)$ shows the inner occurrence of $(b, d)$. Only $(a, c)$ has a proper occurrence, namely $(aeb)b(1)c(fdâ's)$.

**Proposition 1:** Let $(a, b)$ be a singular pair of $x$.

(a) Let $x = x_0 ax_1 bx_2$ be the inner occurrence. Then:

$$a \in x_0 \tau - (x_1 \tau \times) \tau, \quad b \in x_1 \tau - (x_0 ax_1) \tau.$$

(b) Let $x = x_0 ax_1 bx_2$ be a proper occurrence. Then:

$$x_0 \tau \subseteq x_0 \tau \cap x_1 \tau.$$

(c) Let $x_2 \sim y$ and let $x = x_0 ax_1 bx_2$ and $y = y_0 ay_1 by_2$ be inner occurrences. Then:

$$x_0 \tau = y_0 \tau, \quad x_2 \tau = y_2 \tau.$$

(d) Let $x_2 \sim y$ and let $x = x_0 ax_1 bx_2$ be proper and $y = y_0 ay_1 by_2$ be inner. Then $y_1$ has no singular letters of $x$.

**Proof:** (a) If $a \in x_2 \tau$ then $(b, a) \in x \tau_2$ contradicting that $(a, b)$ is singular. If $a \in x_1 \tau$ then the occurrence shown is not inner. If $a \notin x_0 \tau$ then $a$ is a singular letter of $x$, contradicting that $(a, b)$ is a singular pair. The same arguments apply to the claim about $b$.

(b) Let $c \in x_1 \tau$; then $(a, c) \in x \tau_2$. The pair $(a, c)$ cannot be singular because the occurrence of $(a, b)$ as shown is proper. Hence $(c, a) \in x \tau_2$. Since $a \notin (x_1 \times) \tau$, we must have $c \notin x_0 \tau$. Thus $x_1 \tau \subseteq x_0 \tau$, and $x_1 \tau \subseteq x_2 \tau$ follows similarly.

(c) $c \in x_0 \tau$ implies $(c, a) \in x \tau_2 = y_2 \tau$. Hence $c \in y_0 \tau$, and $x_0 \tau \subseteq y_0 \tau$. Similarly $y_0 \tau \subseteq x_0 \tau$ and the claim follows. By symmetry $x_2 \tau = y_2 \tau$.

(d) If $c \in y_1 \tau$ is singular then $(c, a) \notin y \tau_2$. Since $x \tau_2 = y \tau_2$, $c$ must occur exactly once in $x_1$, to satisfy these conditions and the condition that $c$ is a singular letter of $x$. But this contradicts the assumption that $x_0 ax_1 bx_2$ is proper.

**Proposition 2:** Proper occurrences of singular pairs do not overlap, i.e. suppose $x = x_0 ax_1 bx_2$ and $x = y_0 ay_1 by_2$ where the occurrences are proper;
then either $|x_0| \geq |y_0 c y_1 d|$ or $|y_0| \geq |x_0 a x_1 b|$, and $a, b, c, d$ are all distinct.

Proof: Without loss of generality, assume that $|x_0| \leq |y_0|$. Then $c y_1 d \vdash$ to the right of $x_0$. Suppose first the overlap has the form $b = c$ and $x = x_0 a x_1 b y_1 d y_2$. Then $b \not\in (x_0 a x_1) \tau$ because $(a, b)$ is inner as shown and $b \not\in (y_1 d y_2)$ because $(b, d) = (c, d)$ is inner as shown. Hence $b$ is a singular letter, contradicting that $(a, b)$ is a singular pair. Thus this type of overlap cannot occur. Next suppose $x = x_0 a x_1 c x_1 b y_1 d y_2$. We know $a \neq b$ and $c \neq d$. Also $c \neq b$ since $b \not\in (x_0 a x_1 c x_1) \tau$ because $(a, b)$ is inner. Also $c \not\in (x_1 b y_1 d y_2) \tau$ because $(c, d)$ is inner. Hence $(c, b)$ is a singular pair of $x$ contradicting that the occurrence of $(a, b)$ is proper. Again, this type of overlap cannot occur. Thirdly, if $a = c$, then $x = x_0 a x_1 b y_1 d y_2$ and the occurrence of $(a, d)$ cannot be proper. This is a contradiction. Similarly we can’t have $b = d$. Finally, we can’t have $(c, d)$ occur in $x_1$ because the occurrence of $(a, b)$ is proper. Hence, no overlap can occur.

We already know that $a \neq b$, $a \neq c$, $b \neq c$, $b \neq d$, and $c \neq d$. One verifies also that $a \neq d$.

4. ALIGNMENT OF SINGULARITIES

We introduce the following notation to reduce the number of cases that have to be considered. Let:

$$uawbu$$

represent the usual word $uawbu$, with $a, b \in A$, or the word $uau$. The latter case occurs when $w = 1$ and $a = b$. Frequently it is possible to handle both cases by the same arguments, and this notation permits this.

Proposition 3: Let $x = x_0 a x_1 b x_2$ be a proper occurrence of $(a, b)$. Suppose $y_1 \sim x$ and $y = y_0 a y_1 b y_2$ where the occurrence of $(a, b)$ is inner. Then either the occurrence of $(a, b)$ in $y$ is proper or $a y_1 b$ contains exactly one proper occurrence of a singular pair of $x$.

Proof: Suppose $(a, b)$ in $y$ is not proper. By Proposition 1 $(d)$ $y_1$ has no singular letters; hence it must have at least one singular pair. Suppose it has two proper occurrences of singular pairs. By Proposition 2 they do not overlap, so $y$ has the form:

$$y = y_0 a y_1 b c y_1 d y_2 e y_1 f y_1 b y_2,$$

where $(c, d)$ and $(e, f)$ are the two proper occurrences. Now $(d, e) \in y_2 \Rightarrow x_2 z_2$.

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(b, e) ≠ y τ₂ because b is leftmost and e is rightmost; (d, a) ≠ y τ₂ because a is rightmost and d is leftmost.

Thus (e, b) and (a, d) are singular pairs of x. Therefore d ∉ x₀ T, and d ∉ x₁ T because x₀ a x₁ b x₂ shows a proper pair (a, b). Similarly e ∉ x₁ T and e ∉ x₂ T. Hence (d, e) cannot occur in x. This is a contradiction, showing that exactly one singular pair can be proper in y₁.

**Proposition 4:** Let x = x₀ a x₁ b x₂ be a proper occurrence of (a, b) in x. Suppose that x₂ ~ y but y has no proper occurrence of (a, b). By Proposition 3 y has the form y = y₀ a y₁₀ c y₁₁ d y₁₂ b y₂ where the occurrence of (a, b) is inner, either a ≠ c or b ≠ d, and the occurrence of (c, d) is proper. Then:

\[ x = x₀₁ c x₀₂ a x₁ b x₂₁ d x₂₂ \]

where the occurrence of (c, d) is inner.

**Proof:** Observe that (a, d) ∈ y τ₂ but (d, a) ≠ y τ₂ because a is rightmost and d is leftmost. Hence (a, d) ∈ x₁ T and (d, a) ∉ x₂ T. Thus d ∉ x₀ T. Also d ∉ x₁ T because the singular pair (a, d) would appear in a x₁ b and the latter is assumed to be proper. Thus d ∈ (b x₂) T and x = x₀ a x₁ b x₂₁ d x₂₂, where d ∉ x₂₁ T. Similarly, (c, b) ∈ x τ₂, (b, c) ∉ x T and x₀ a = x₀₁ c x₀₂ a, giving the desired form for x.

**Lemma 1:** Let x₂ ~ y, where x and y are coterminal paths in a graph. Then there exists y' ~ y such that a proper occurrence of a singularity exists in x iff it exists in y'. Further, if x = x₀ a x₁ b x₂ where (a, b) is proper, then y' = y₀ a x₁ b y₂.

**Proof:** (i) If x = xₑ a xₑ where e is a singular letter, we must have y = yₑ e yₑ, since the occurrence of a singular letter is always proper.

(ii) Suppose x = x₀ a x₁ b x₂ and y = y₀ a y₁ b y₂ where both occurrences are proper. By Proposition 1(b), x₁ T ⊂ x₀ T ∩ x₂ T and x₀ T = y₀ T, x₂ T = y₂ T by Proposition 1(c). Thus x₁ T ⊂ y₀ T ∩ y₂ T. Also y₁ T ⊂ y₀ T ∩ y₁ T. Since x₁ and y₁ are coterminal paths, we can apply Rule (1a):

\[ y = (y₀ a) y₁ (b y₂) \sim (y₀ a) x₁ (b y₂) = y'. \]

(iii) Suppose y is as above, but the occurrence of (a, b) is not proper. Then, by Proposition 3:

\[ y = y₀ a y₁₀ c y₁₁ d y₁₂ b y₂ \]

where (c, d) is proper and (a, b) is inner and either a ≠ c or d ≠ b or both.

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Then, by Proposition 4:

\[ x = x_{01} c x_{02} a x_1 b x_{21} d x_{22}, \]  

(4)

where \((a, b)\) is proper, \((c, d)\) is inner and either \(a \neq c\) or \(b \neq d\) or both.

**Case 1: \(a \neq c, b = d\)**

We have the following factorizations:

\[ x = x_{01} c x_{02} a x_1 b x_2, \]
\[ y = y_{01} a y_{10} c y_{11} b y_2. \]

Let \(u = y_{10} c y_{11} b y_2\), so that \(y = y_0 a u\) where \(a\) is rightmost. Then \(a \notin u\) and \((x_{02} a) \tau \notin u \tau\). However, \((x_{02} a) \tau \subset y \tau\) because \(x_2 \sim y\) implies \(x_2 \tau = y \tau\). Therefore there must exist precisely one suffix \(w = e x_{02} a u\) of \(y\) such that \((x_{02} a) \tau \subset w \tau\) but \((x_{02} a) \tau \notin (e x_{02} a u) \tau\), where \(y_{02} a\) denotes \(y_{02} a\) when \(e \neq a\) and \(y_{02} a = 1\), when \(e = a\). Note that \(e \notin (e x_{02} a u) \tau\) and also that \(e\) must be a letter of \(x_{02} a\); let \(x_{02} a = x_{02} e x_{02} a\), where \(e \notin x_{02} \tau\). Then:

\[ x = x_{01} c x_{02} e x_{02} a x_1 b x_2, \]
\[ y = y_{01} e y_{02} a y_{10} c y_{11} b y_2 = y_{01} w. \]

Consider the loop \(h = e y_{02} a y_{10} c x_{02} \). We claim that this loop can be inserted after \(y_{01}\) in \(y\) by using Rule (1a). For we have \((e y_{02} a y_{10} c) \tau \subset w \tau\) by the definition of \(w\) above. Also \(x_{02} \tau \subset (x_{02} a) \tau \subset w \tau\). Thus \(h \tau \subset w \tau\).

Next we must verify that \(h \tau \subset y_{01} \tau\). By construction \(e\) is rightmost in \(y\). Thus \(f \in (c x_{02}) \tau\) implies \((f, e) \in x_2 \tau = y_2 \tau\) and \(f \in y_{01} \tau\). Hence \(c x_{02} \tau \subset y_{01} \tau\). In fact we have \((x_0 \tau) \tau \subset y_{01} \tau\) by the same argument. Now \(f \in (c y_{10} c y_{11} b y_2) \tau\) implies \((f, c) \in y_2 \tau = x_2 \tau\) and \(f \in x_{01} \tau\) because \(c\) is rightmost in \(x\) as shown. Thus \(f \in y_{01} \tau\). Altogether, \(h \tau \subset y_{01} \tau\). Inserting two copies of the loop \(h\) we have:

\[ y = y_{01} e y_{02} a y_{10} c y_{11} b y_2 \]
\[ \sim y_{01} (y_{10} c x_{02} e y_{02} a y_{10} c y_{11} b y_2)^2 y_{02} a y_{10} c y_{11} b y_2 \]
\[ = y_{01} e y_{02} a (y_{10} c x_{02} e y_{02} a)^2 y_{10} c y_{11} b y_2. \]

Let \(z_1 = y_{01} e y_{02} a, \ p = y_{10} c, \ q = x_{02} e y_{02} a, \ z = y_{11}, \) and \(z_2 = b y_2\). Then:

\[ y \sim z_1 (pq)^2 p z_2. \]

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We now show that $z \tau \subset z_1 \tau \cap z_2 \tau$. In fact, $f \in y_1 \tau$ implies $(c, f) \in y \tau_2$ and so $(f, c)$ in $y \tau_2 = x \tau_2$ because $(c, b) = (c, d)$ is proper in $y$. Thus $f \in x_1 \tau$, and we have $f \in x_1 \tau$. Therefore $z \tau \subset z_1 \tau$. Similarly $f \in y_1 \tau$ implies $(f, b) \in y \tau_2$ and $(b, f) \in y \tau_2$. Hence $f \in y_2 \tau$ and $z \tau \subset z_2 \tau$.

Let $x = x_1$. Then $x_1 \tau \subset z_1 \tau \cap z_2 \tau$ by similar arguments. We are now in a position to apply Rule (I b):

$$y \sim z_1 (pq)^2 \rho z \sim z_1 (pq)^2 z_2$$

$$= y_{01} e_{y_{02}} d (x_{10} c x_{02} e y_{02} d)^2 x_1 b y_2$$

$$= [y_{01} (e y_{02} a y_{10} c x_{02})^2 e y_{02}] a x_1 b y_2$$

$$= y'_1 a x_1 b y'_2 = y'$$

which has the desired form. We can also write:

$$y' = y_{01} e y_{02} g^2 a x_1 b y_2 = y_0 g^2 a x_1 b y_2,$$

where $g = a y_{10} c x_{02} e y_{02}$. Recall that proper singularities do not overlap. In $y = y_0 a y_{10} c y_{11} b y_2$ we have the proper singularities in $y_0 a y_{10}$ and in $y_2$ and the pair $(c, b)$. By Proposition 3 the segment $a y_{10} c y_{11} b$ has only one proper singularity; hence there are none in $a y_{10}$. Now in $y'$ we have the proper singularities of $y_0 a y_{10}$ and $y_2$ and the pair $(a, b)$ which replaced $(c, b)$. The segment $g^2$ is free of singularities, since each pair $(f, f') \in g \tau \times g \tau$ appears at least twice in $g^2$ if $f \neq f'$, and $g^2$ can't have any singular letters. This leaves the possibility that there is a proper singularity in $y_0 g$ of the type $f \in y \tau$, $f' \in g \tau$. But $g \tau \subset y_{01} \tau \subset y_0 \tau$. Hence either $(f', f) \in y_0 \tau_2$ and $(f, f')$ is not singular, or $(f', f) \in y_0 \tau_2$ and the singularity in $y_0 g$ was not proper. Thus $y'$ has only the proper singularities of $y$ with $(c, d)$ replaced by $(a, b)$.

**Case 2: $a = c, b \neq d$**

This follows by left-right symmetry from Case 1. This time a loop is inserted on the right side and Rule (I c) is applied.

**Case 3: $a \neq c, b \neq d$**

Proceed as in Case 1 inserting first the left loop, then the right loop, and apply Rule (I).

In all cases of (iii) we can transform $y$ into $y'$ in such a way that the proper singularities of $y'$ are the same as those of $y$ except that $(c, d)$ has been replaced by $(a, b)$. Now consider two words $x, y \in A^*$ such that $x_2 \sim y$.

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each singular letter of $x$ must also be a singular letter of $y$ and vice versa. Also, if $(a, b)$ has a proper occurrence in $x$ then either $(a, b)$ is also proper in $y$, or $(a, b)$ occurs in $y$ with another proper pair $(c, d)$, as in Propositions 1 and 4. As shown above, we can find $y'$ such that $y' \sim y$ and the singularities of $y'$ are those of $y$, with the exception that $(c, d)$ has been replaced by $(a, b)$. By repeating this process we find $y' \sim y$ such that $y'$ has exactly the same singularities as $x$. It is easily verified that these singularities must appear in $y'$ in the same order as in $x$. Thus we may assume at this point that $x$ and $y$ have the same singularities and that they have the form:

\[
x = x_0 s_1 x_1 s_2 \ldots s_m x_m
\]

\[
y = y_0 s_1 y_1 s_2 \ldots s_m y_m
\]

where $m \geq 0$, $x_i$, $y_i$, $i = 0, \ldots, m$, do not have any singularities of $x$ and either $s_i = e$, $e \in A$, or $s_i = aw_i b$ is a proper singular pair of $x$.

5. SEGMENTS BETWEEN SINGULARITIES

Refer to the factorizations of $x$ and $y$ above that show all the proper singularities. In this section we will show that the segments $y_i$ between proper singularities can be replaced by the segments $x_i$ by using only Rule (1). The main result here is Lemma 2, but we need several preliminary results first.

**Proposition 5**: Let:

\[
x = x_1 x_2 x_3 = (x_0 s_1 \ldots x_i s_i) x_{i+1} (s_{i+1} x_{i+2} \ldots s_m x_m)
\]

$i \geq 0$, $m \geq 0$, where $x_1 = x_0 s_1 \ldots x_i s_i$, $x_2 = x_{i+1}$, and $x_3 = (s_{i+1} x_{i+2} \ldots s_m x_m)$ and let:

\[
y = y_1 y_2 y_3 = (y_0 s_1 \ldots y_i s_i) y_{i+1} (s_{i+1} y_{i+2} \ldots s_m y_m)
\]

be similarly defined, where $x_2 \sim y_2$, $x$ and $y$ are coterminial, and $x$ and $y$ have the same proper singularities. Then $x_2$ and $y_2$ are coterminial and

\[
(x_1 x_2) \tau = (y_1 y_2) \tau, \quad (x_2 x_3) \tau = (y_2 y_3) \tau.
\]

**Proof**: If $x$ has no proper singularities then $x_2 = x$ and $y_2 = y$ and the claims easily follow. If $x$ has exactly one singularity then either $x_1 = 1$, $x_2 = x_0$, $x_3 = s_1 x_1$, or $x_1 = x_0 s_1$, $x_2 = x_1$, and $x_3 = 1$. In the first case $y_1 = 1$, $y_2 = y_1$, and $y_3 = x_1$. Again the claim is easily verified here, and the second case is symmetric. The general case follows easily with the aid of Proposition 1(c).
Proposition 6: Let \( x \in A^* \) have the factorization:

\[
x = x_1 x_2 x_3 = x_1 x_{21} a x_{22} x_3,
\]

where \( x_2 = x_{21} a x_{22}, a \in A, \) and \( a \notin (x_1 x_{21})^\tau \). If \( x_2 \) has no singularities of \( x \), then:

\[
(x_{21} a)^\tau \subseteq (x_{22} x_3)^\tau.
\]

Proof: Since \( a \) appears in \( x_2 \) and \( x_2 \) has no singularities of \( x \), we have \((a, a) \in x_\tau \). Because \( a \notin (x_1 x_{21})^\tau \), we must have \( a \notin (x_{22} x_3)^\tau \). Also \( e \in x_{21}^\tau \) implies \((e, a) \in x_\tau \). Since \( x_3 \) has no singularities of \( x \), we have \((a, e) \in x_{21}^\tau \) and \( e \in (x_{22} x_3)^\tau \). Thus \((x_{21} a)^\tau \subseteq (x_{22} x_3)^\tau \).

Proposition 7: Let \( x, y \in A^* \) have the factorizations:

\[
x = x_1 x_2 x_3 = x_1 x_{21} a x_{22} x_3,
\]

\[
y = y_1 y_2 y_3 = y_1 y_{21} a y_{22} y_3,
\]

where \( x_2 \) and \( y_2 \) have no singularities of \( x \), and \( x_2 = x_{21} a x_{22}, y_2 = y_{21} a y_{22}, a \in A, \ a \notin (x_1 x_{21})^\tau \cup (y_1 y_{21})^\tau \). Then \((x_{22} x_3)^\tau = (y_{22} y_3)^\tau \) implies \((x_{22} x_3)^\tau = (y_{22} y_3)^\tau \) by Proposition 6. Similarly \((y_{22} y_3)^\tau = (y_{22} y_3)^\tau \) and the claim follows.

Let \( x, y \in A^* \) be such that \( x \tau = y \tau \) and let \( B \) be a given subset of \( x \tau \). Let \( \bar{x} \) and \( \bar{y} \) be prefixes of \( x \) and \( y \) respectively. The pair \((\bar{x}, \bar{y})\) is called a \( B \)-pair iff:

\[
\bar{x} \tau = \bar{y} \tau \supseteq B.
\]

Let \( P_B(x, y) \) be the set of all \( B \)-pairs of \( x \) and \( y \). This set is nonempty since \((x, y) \in P_B(x, y)\). Define the binary relation \( \leq \) on \( P_B(x, y) \) by:

\[
(x_1, y_1) \leq (x_2, y_2) \text{ iff } \left| x_1 \right| \leq \left| x_2 \right| \text{ and } \left| y_1 \right| \leq \left| y_2 \right|.
\]

One verifies that \( \leq \) is a partial order on \( P_B(x, y) \).

Proposition 8: \( P_B(x, y) \) has a unique minimal element with respect to \( \leq \).

Proof: Because \( P \) is finite it suffices to show that for all \( p_1 = (x_1, y_1), p_2 = (x_2, y_2) \) in \( P_B(x, y) \) there exists \( \bar{p} = (\bar{x}, \bar{y}) \in P_B(x, y) \) such that \( \bar{p} \leq p_1 \) and \( \bar{p} \leq p_2 \). If \( p_1 \leq p_2 \), let \( \bar{p} = p_1 \). If \( p_1 \leq p_2 \), let \( \bar{p} = p_2 \). Now suppose neither \( p_1 \leq p_2 \) nor \( p_2 \leq p_1 \). Suppose also that \( \left| x_1 \right| > \left| x_2 \right| \). Then, since \( p_1 \notin p_2 \), we must have \( \left| y_1 \right| < \left| y_2 \right| \). Now:

\[
x_1 \tau \subseteq x_1 \tau = y_1 \tau \subseteq y_2 \tau = x_2 \tau.
\]

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Let $\overline{p} = (x_2, y_1)$. Then $\overline{p}$ is a B-pair and $\overline{p} \leq p_1, \overline{p} \leq p_2$. Similarly, if $|x_1| < |x_2|$ then $|y_1| > |y_2|$. Let $\overline{p} = (x_1, y_2)$; then $\overline{p}$ is the required B-pair. Finally the case $|x_1| = |x_2|$ cannot occur, for then either $p_1 \leq p_2$ or $p_2 \leq p_1$.

**Lemma 2:** Let $x$ and $y$ be coterminant paths such that $x \sim y$ and suppose that $x$ and $y$ have the factorizations:

$$x = x_1 x_2 x_3, \quad y = y_1 y_2 y_3,$$

where $x_2$ and $y_2$ are coterminant and do not contain any singularities of $x$ and:

$$x_1 \tau = y_1 \tau, \quad x_3 \tau = y_3 \tau,$$

$$(x_1 x_2) \tau = (y_1 y_2) \tau, \quad (x_2 x_3) \tau = (y_2 y_3) \tau.$$

Then $y \sim y_1 x_2 y_3$.

**Proof:** The proof proceeds by induction on $|x_2| + |y_2|$.

**Basis:** $|x_2| + |y_2| = 0$

Here $x_2 = y_2 = 1$ and $y = y_1 1 y_3 \sim y_1 x_2 y_3$.

**Induction Step:** $|x_2| + |y_2| > 0$

We assume that the lemma holds for all cases where $|x_2| + |y_2| \leq k$. Suppose now that $|x_2| + |y_2| = k + 1$. The proof will be decomposed into several cases.

**Case 1:** $x_2 \tau \subset x_1 \tau$ and $x_2 \tau \subset x_3 \tau$

Here $y_2 \tau \subset (y_1 y_2) \tau = (x_1 x_2) \tau = x_1 \tau = y_1 \tau$. Similarly $y_2 \tau \subset y_3 \tau$. Also $x_2 \tau \subset y_1 \tau \cap y_3 \tau$. By Rule (1 a):

$$y = y_1 y_2 y_3 \sim y_1 x_2 y_3.$$

**Case 2:** $x_2 \tau \not\subset x_1 \tau$

Note that $y_2 \tau \not\subset y_1 \tau$; otherwise:

$$x_2 \tau \subset (x_1 x_2) \tau = (y_1 y_2) \tau = y_4 \tau = x_1 \tau,$$

which is a contradiction. Let $a$ be the first letter of $x_2$ from the left that does not appear in $x_1$. Similarly let $b$ be the first letter of $y_2$ from the left that is not in $y_1$. Then $x_2 = ax_2, y_2 = y_2 b y_2$ and

$$x = x_1 x_2 a x_2 x_3, \quad \text{where} \quad a \notin (x_1 x_2) \tau = x_1 \tau, \quad (5)$$

$$y = y_1 y_2 b y_2 y_3, \quad \text{where} \quad b \notin (y_1 y_2) \tau = y_1 \tau. \quad (6)$$

We consider next two subcases.
Case 2.1: \( a = b \)

Here we have:

\[
y = y_1 y_2 \alpha y_2 y_3, \quad \text{where} \quad a \notin (y_1, y_{21}) \tau = y_1 \tau,
\]

and \( x \) is as in (5). Now \( x_{21} \) and \( y_{21} \) are coterminial and \( x_{21} \tau, y_{21} \tau \subset y_1 \tau \).

By Proposition 6, \( y_{21} \tau \subset (y_{22} y_3) \tau \). By Propositions 6 and 7, \( x_{21} \tau \subset (x_{22} x_3) \tau = (y_{22} y_3) \tau \). By Rule (1 a):

\[
y = (y_1) (y_{21}) (a y_{22} y_3) \cap (y_1) (x_{21}) (a y_{22} y_3) = y'.
\]

Now let \( x'_1 = x_1 x_{21} a, x'_2 = x_{22}, \) and \( x'_3 = x_3 \). Then:

\[
x = x'_1 x'_2 x'_3 = (x_1 x_{21} a) (x_{22}) (x_3).
\]

Similarly, let \( y'_1 = y_1 x_{21} a, y'_2 = y_{22}, \) and \( y'_3 = y_3 \). Then:

\[
y = y'_1 y'_2 y'_3 = (y_1 x_{21} a) (y_{22}) (y_3).
\]

We verify the 4 conditions of the lemma:

(i) \( x'_1 \tau = (x_1 x_{21} a) \tau = (y_1 x_{21} a) \tau = y'_1 \tau \).

(ii) \( x'_2 \tau = (x_2 x_3) \tau = (y_1 y_2') \tau = (y'_1 y'_2) \tau \).

(iii) \( x'_3 \tau = (y_3) \tau = (y'_3) \tau \).

(iv) \( x'_2 x'_3 = (x_{22} x_3) \tau = (x_{22} x_3) \tau = (y_{22} y_3) \tau = (y'_2 y'_3) \tau \) by Proposition 7.

Note that \( x'_2 \) is a proper factor of \( x_2 \) and \( y'_2 \) is a proper factor of \( y_2 \).

Hence \( x'_2 \) and \( y'_2 \) do not contain any singularities of \( x \). Evidently \( |x'_2| + |y'_2| < |x_2| + |y_2| \) and we can apply the induction hypothesis:

\[
y' = y'_1 y'_2 y'_3 \cap y'_1 y'_2 y'_3 = y_1 x_{21} a y_{22} y_3 = y_1 x_2 y_3.
\]

Altogether \( y' \cap y_1 x_2 y_3 \) and the induction step goes through in this case.

Case 2.2: \( a \neq b \)

Refer to (5) and (6). Since \( b \in (y_1, y_2) \tau = y_1 \tau = (x_1 x_2) \tau = x_1 \tau \) we must have \( b \in x_{22} \tau \). Similarly \( a \in y_{12} \tau \) and:

\[
x = x_1 x_2 x_3 = x_1 (x_{21} a x_{22}) x_3 = x_1 x_{21} a (s_1 b s_2) x_3,
\]

where \( x_{22} = s_1 b s_2 \) and \( b \notin (x_1 x_{21} a s_1) \tau \), and

\[
y = y_1 y_2 y_3 = y_1 (y_{21} b y_{22}) y_3 = y_1 y_{21} b (t_1 a t_2) y_3,
\]

where \( y_{22} = t_1 a t_2 \) and \( a \notin (y_1, y_{21} b t_1) \tau \). In other words the leftmost appearances of \( b \) in \( x \) and \( a \) in \( y \) are shown.

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Let \((as_1) \tau \cup (bt_1) \tau = B\). The prefixes \(x_1 x_2\) of \(x_1 x_2\) and \(y_1 y_2\) of \(y_1 y_2\) satisfy:

\[(x_1 x_2) \tau = (y_1 y_2) \tau \supset B.\]

Thus \((x_1 x_2, y_1 y_2)\) is a B-pair. By Proposition 8, there exists a minimal B-pair \((\bar{x}, \bar{y})\). Since \(b \in B\) and \(b \notin (x_1 x_2, as_1) \tau\) we have:

\[|x_1 x_2 as_1 b| \leq |\bar{x}| \leq |x_1 x_2|,\]  \hspace{1cm} (13)

Similarly:

\[|y_1 y_2 bt_1 a| \leq |\bar{y}| \leq |y_1 y_2|,\]  \hspace{1cm} (14)

Let \(c\) be the last letter of \(\bar{x}\) and \(d\) the last letter of \(\bar{y}\), and let \(\bar{x} = pc\) and \(\bar{y} = qd\). We claim first that \(c \neq d\). Note that \(c \neq p\), otherwise the pair \((p, \bar{y})\) would be a shorter B-pair. Similarly \(d \neq q\). Assume now that \(c = d\). If \(c \notin \bar{x}\) then \((p, q)\) is a B-pair, contradicting the assumption that \((pc, qd)\) is minimal. Thus \(c \in B = (as_1, bt_1) \tau\). Since \(|x_1 x_2 as_1 b| \leq |pc|\) and \(c \notin \bar{x}\), the condition \(c \notin (as_1 b) \tau\) implies \(c = b\). But then \(c \in (y_1 y_2 bt_1) \tau\) and \(y_1 y_2 bt_1\) is a proper prefix of \(\bar{y}\). This implies \(c \notin q\) which is a contradiction. Hence we cannot have \(c \notin (as_1 b) \tau\) and we must have \(c \in \bar{t}_1\). This is again a contradiction of the fact that \(c \neq d\).

From (13) and (11) it is clear that either \(c = b\) or \(c \neq b\) and \(c \in s_2\). Both cases can be handled by the notation:

\[pc = x_1 x_2 as_1 bs_2,\]  \hspace{1cm} (15)

For if \(c = b\), let \(s_{21} = 1\). Otherwise let \(s_{21}\) be the shortest prefix of \(s_2\) that ends in \(c\). In either case let \(s_2 = s_{21} s_{22}\). Similarly:

\[qd = y_1 y_2 bt_1 at_{21},\]  \hspace{1cm} (16)

where \(t_2 = t_{21} t_{22}\) and \(t_{21} = 1\) if \(d = a\), and \(t_{21}\) is the shortest prefix of \(t_2\) that ends in \(d\). Otherwise, now let:

\[f = as_1 bs_{21},\]
\[g = bt_1 at_{21}.\]

We now arrive at the decompositions of \(x\) and \(y\):

\[x = x_1 x_2 x_3 = x_1 x_2 x_3 = x_1 x_2 x_3 as_1 b s_{21} x_3 = x_1 x_2 x_3,\]  \hspace{1cm} (17)
\[y = y_1 y_2 y_3 = y_1 y_2 y_3 b t_1 y_3 = y_1 y_2 bt_1 y_3 = y_1 y_2 b t_1 at_2 y_3 = y_1 y_2 b t_1 at_2 t_{21} y_3 = q d t_{22} y_3.\]  \hspace{1cm} (18)

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Consider next where $c$ can appear in $y$. Since $c \in (pc) \tau = (qd) \tau$, we must have $c \in (y_1 y_{21} bt_1 at_{21}) \tau$. If $c \in (y_1 y_{21}) \tau$ then $c \in x_1 \tau$ and $c \in \rho \tau$ which is a contradiction. Hence $c \in (bt_1 at_{21}) \tau = g \tau$. Similarly $d \in (as_1 bs_{21}) \tau = f \tau$. Let:

\[ f = as_1 bs_{21} = u_1 dv_2 c, \quad \text{where} \quad d \notin u_2 \tau, \quad (19) \]
\[ g = bt_1 at_{21} = v_1 cv_2 d, \quad \text{where} \quad c \notin v_2 \tau. \quad (20) \]

In other words we take the rightmost appearances of $d$ in $f$ and $c$ in $g$. We now have the factorizations illustrated in Figure 2. Of necessity, the figure shows a particular case and should only be used as a visual aid.

We will deal with the factorization:

\[ x = x_1' x_2' x_3' = (x_1 x_{21} f)(s_{22})(x_3), \quad (21) \]

where $x_1' = x_1 x_{21} f$, $x_2' = s_{22}$, $x_3' = x_3$. We begin with:

\[ y = y_1 y_{21} g t_{22} y_3 \]

and we will show that $y \sim y'$ where:

\[ y' = y_1' y_2' y_3' = (y_1 x_{21} f)(v_2 d t_{22})(y_3), \quad (22) \]

where $y_1' = y_1 x_{21} f$, $y_2' = v_2 d t_{22}$, and $y_3' = y_3$. The proof is given in Lemma 3 below. Assuming this result we next show that all the conditions of Lemma 2 apply to (21) and (22).

First, $x_3' = s_{22}$ is a proper factor of $x_2$ and $y_2' = v_2 d t_{22}$ is a proper factor of $y_2$. Hence $x_2'$ and $y_2'$ contain no singularities of $x$. Second, $x_2'$ and $y_2'$ are coterminant. Third, $y \sim y'$ (Lemma 3) implies $y_2 \sim y_2'$ and so $x_2 \sim y'$. Finally, we verify the four conditions on the alphabets of the factors:

(i) $x_1' \tau = (x_1 x_{21} f) \tau = (y_1 x_{21} f) \tau = y_1' \tau$.

(ii) $(x_1' x_2') \tau = (x_1 x_2) \tau = (y_1 y_2) \tau = (qd) \tau \cup t_{22} \tau$

\[ = (pc) \tau \cup t_{22} \tau = (x_1 x_{21} f) \tau \cup t_{22} \tau \]

\[ = (y_1 x_{21} f) \tau \cup t_{22} \tau = (y_1 x_{21} f) \tau \cup (v_2 d) \tau \cup t_{22} \tau. \]

because $(v_2 d) \tau \subset (y_1, y_3) \tau$. Therefore:

\[ (x_1' x_2') \tau = (y_1 x_{21} f v_2 d t_{22}) \tau = (y_1' y_2') \tau. \]

(iii) $x_3' \tau = x_3 \tau = y_3 \tau = y_3' \tau$.

(iv) Since $y_1'$ ends in $f$ which ends in $c$, $c \in (y_2' y_3') \tau$ implies
(c, e) ∈ y' τ₂ = x τ₂. Hence e ∈ (s₂₂ x₃) τ, because c ≠ p τ. Therefore
(y₁' y₃') τ ⊆ (x₂' x₃') τ.

Conversely:

(x₂' x₃') τ = (t₂₂ x₃) τ ⊆ (x₂ x₃) τ = (y₁ y₃) τ = (y₂₁ g t₂₂ y₃) τ.

By Proposition 6 applied to the letter d in g, (y₂₁ g) τ ⊆ (t₂₂ y₃) τ. Hence
(x₂' x₃') τ ⊆ (t₂₂ y₃) τ ⊆ (y₁ y₃') τ. Thus (x₂' x₃') τ = (y₁' y₃') τ.

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Now all the conditions of Lemma 2 are satisfied. Since $|x_2| + |y_2| < |x_2| + |y_2|$, the induction hypothesis applies and

$$y' = y'_1 y'_2 y'_3 \sim y'_1 x'_2 y'_3 = y'_1 x_2 y_3,$$

Therefore $y \sim y' \sim y_1 x_2 y_3$ as claimed, and the induction step goes through.

Case 3: $x_2 \tau \notin \tau x_3$

This follows from Case 2 by left-right duality.

Since the induction step goes through in all cases, the lemma holds.

**Lemma 3:** Let $x$, $y$, and $y'$ be defined as in the proof of Lemma 2. Then $y \sim y'$.

**Proof:** (a) We first show that the graph consisting of the edges in $C = f \tau \cup g \tau$ is strongly connected. Since the node $b \omega$ is connected to $a \alpha = f \alpha$ by the path $t_1$, all the nodes in the path $as_1 b$ are connected to $f \alpha$. Let $s_{21} = s_{21}' s_{21}''$ where $s_{21}'$ is the longest prefix of $s_{21}$ that is connected to $f \alpha$. Similarly, $a \alpha$ is connected to $b \alpha = g \alpha$ by $s_1$. Let $t_{21} = t_{21}' t_{21}''$ where $t_{21}'$ is the longest prefix of $t_{21}$ connected to $g \alpha$ (see Fig. 3).

![Diagram](image_url)

Figure 3.

Now $s_{21}''$ cannot have any edges in common with $as_1 b s_{21}'$ or $b t_{21} a t_{21}'$. Otherwise the $\omega$ end of the common edge could be connected to $f \alpha$. Hence:

$$s_{21}'' \tau \cap (b t_{21} a t_{21}') \tau = \emptyset.$$

Also, $(pc) \tau \Rightarrow (qd) \tau$, i.e.:

$$(x_1 x_2 as_1 b s_{21}'' \tau \cup s_{21}' \tau = (y_1 y_2 b t_{21} a t_{21}) \tau = (y_1 y_2 x_{21}) \tau \cup (b t_{21} a t_{21}') \tau.$$

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Consequently we have:

\[(x_1 x_{21} as_1 b s_{21}) \tau \supset (y_1 y_{21} b t_1 a t'_{21}) \tau.\]

Similarly the reverse inclusion holds:

\[(x_1 x_{21} as_1 b s_{21}) \tau = (y_1 y_{21} b t_1 a t'_{21}) \tau \supset B = (a s_1) \tau \cup (b t_1) \tau.\]

Therefore \((x_1 x_{21} as_1 b s_{21}, y_1 y_{21} b t_1 a t'_{21})\) is a \(B\)-pair. However \((pc, qd)\) is a minimal \(B\)-pair. Hence we must have \(s'_{21} = s_{21}, \ t'_{21} = t_{21}, \ f \omega \) is connected to \(f \alpha\) and \(g \omega\) is connected to \(g \alpha\). Hence the graph is strongly connected since \(f\) and \(g\) have a common edge.

(b) In view of (a) there exists paths \(h\) and \(k\) such that:

\[h \alpha = f \omega, \quad h \omega = f \alpha, \quad h \tau \subset C,\]
\[k \alpha = g \omega, \quad k \omega = g \alpha, \quad k \tau \subset C.\]

Let \(f' = u_2 ch\) and \(g' = v_2 dk\). Then \(f' g' g\) is a loop about the vertex \(d \omega = g \omega\) and \(f' g' g \subset C\). Now:

\[y = y_1 y_{21} g t_{22} y_3\]
\[\sim (y_1 y_{21} g) (f' g' g) t_{22} y_3\]

by (1 a), because \((y_1 y_{21} g) \tau = (qd) \tau = (te) \tau \supset C\), and \(C \subset (t_{22} y_3) \tau\) by Proposition 6. Thus:

\[\sim y_1 y_{21} g (f' g' g) t_{22} y_3\]
\[= y_1 y_{21} (g f') ((f' g') (f' g')) t_{22} y_3\]
\[= [y_1] (y_{21} (g f')) [((f' g') (f' g'))] [f' g' t_{22} y_3].\]

Now Rule (1 c) can be applied, yielding:

\[y \sim y_1 x_{21} (f' g' g')^2 f' g' t_{22} y_3\]

where we have replaced \(y_{21} g f'\) by \(x_{21}\). The alphabet conditions on \(x_{21}\) and \(y_{21}\) are easily verified. Thus:

\[x_2 t_{22} y_3,\]

\[y \sim y_1 x_{21} (f' g' g')^2 f' g' t_{22} y_3\]
\[= y_1 x_{21} f' g' (f' g')^2 g t_{22} y_3\]
\[\sim y_1 x_{21} f' g t_{22} y_3, \quad \text{by Rule (1 a)}\]
\[= y_1 x_{21} f v_2 d (kv_1 cv_2 d) t_{22} y_3\]
\[\sim y_1 x_{21} f v_2 t_{22} y_3, \quad \text{by Rule (1 a)}\]
\[= y'.\]
Hence the lemma holds.
This concludes the proof of Lemmas 2 and 3. By combining Lemmas 1 and 2 we have the theorem.

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