

RESEARCH ARTICLE

A GENERALIZATION OF FINITENESS

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Communicated by M.A. Arbib

We explore an analogy between the family  $\beta_1$  of finite/cofinite languages and the family  $\gamma_1$  of languages whose syntactic monoids are  $J$ -trivial. It is shown that (a)  $J$ -trivial monoids, (b)  $L$ -trivial monoids, (c)  $R$ -trivial monoids, and (d) a recently studied family, that we call  $K$ , of aperiodic monoids are natural generalizations of the families of syntactic semigroups of (a) finite/cofinite languages, (b) definite languages, (c) reverse definite languages, and (d) generalized definite languages, respectively. In the case of alphabets of one and two letters, the languages corresponding to the family  $K$  of monoids are characterized, illustrating the above-mentioned analogy explicitly.

1. INTRODUCTION

Let  $A$  be a finite, non-empty alphabet and  $A^*$  the free monoid generated by  $A$ , with unit element  $1$ . For any family  $X$  of languages over  $A$  (i.e. subsets of  $A^*$ ) let  $\mathcal{X}M$  be the smallest family of languages over  $A$  containing  $X \cup \{1\}$  and closed under concatenation. Similarly let  $\mathcal{X}B$  be the smallest family containing  $X$  and closed under finite unions and complementation with respect to  $A^*$ . Thus  $\mathcal{X}M$  and  $\mathcal{X}B$  are the monoid and Boolean algebra (respectively) generated by  $X$ .

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\* This work was done partly at the University of Paris VI and VII under the scientific exchange program between Canada and France, and partly at the Institut für Rechner- und Programmstrukturen, Gesellschaft für Mathematik und Datenverarbeitung mbH. Bonn, Germany.

Define  $L = \{\{a\} \mid a \in A\}$ ,  $W = LM$ ,  $F = \{X \subset A^* \mid X \text{ is finite}\}$  and  $C = \{X \subset A^* \mid \bar{X} \in F\}$ , where  $\bar{X} = A^* - X$ . One easily verifies [CO-BR] that  $LMB = F \cup C$ . Let  $\beta_1$  be the family  $F \cup C = LMB$  of finite/cofinite languages.

One can generalize  $\beta_1$  as follows. Let  $L_{\oplus} = L \cup \{a^+ \mid a \in A\}$ ; let  $W_{\oplus} = L_{\oplus} M$ ; let  $F_{\oplus} = \{X \subset A^* \mid X \text{ is a finite union of languages in } W_{\oplus}\}$  and let  $C_{\oplus} = \{X \mid \bar{X} \in F_{\oplus}\}$ . Again it can be shown that  $L_{\oplus} MB = F_{\oplus} \cup C_{\oplus}$  [BRZ]. Furthermore, if  $\text{card } A$  (the cardinality of  $A$ ) is  $\leq 2$ , the family  $L_{\oplus} MB$  coincides with the family  $\gamma_1$  of languages over  $A$  whose syntactic monoids are  $J$ -trivial.

The analogy between  $\beta_1 = LMB$  and  $\gamma_1 = L_{\oplus} MB$  for a two-letter alphabet is generalized in this paper.

## 2. SYNTACTIC SEMIGROUPS AND MONOIDS

For  $X \subset A^*$  the syntactic congruence  $\equiv_X$  is defined as follows. For  $x, y \in A^*$ ,  $x \equiv_X y$  iff for all  $u, v \in A^*$ ,  $uxv \in X$  iff  $uyv \in X$ . The quotient monoid  $M_X \triangleq A^*/\equiv_X$  is called the syntactic monoid of  $X$ . (The symbol  $\triangleq$  means "is by definition".)

With each syntactic monoid  $M_X$  we associate the syntactic morphism  $\mu_X: A^* \rightarrow M_X$ ; this is the natural morphism of  $A^*$  onto  $M_X$  that assigns to each word in  $A^*$  its syntactic congruence class. For brevity we will say that " $M_X$  is a monoid over  $A$  with morphism  $\mu_X$ ".

The syntactic semigroup  $S_X$  of  $X$  is the quotient semigroup  $S_X \triangleq A^+/\equiv_X$ . For certain purposes it is necessary to use  $S_X$  rather than  $M_X$ . The basic difference is that, if  $S_X$  has a unit element  $1_S$ , then there exists a non-empty word  $x$  such that  $x\mu_X = 1_S$ . This information is not available in  $M_X$ , since one always has  $1\mu_X = 1_M$ . When discussing properties of languages that are reflected in syntactic semigroups but not in syntactic monoids, we assume that each language is a subset of  $A^+$ , and that complementation is with respect to  $A^+$  and not  $A^*$ .

For terminology and notation not defined in this paper refer to [CL-PR].

## 3. GENERALIZED DEFINITE LANGUAGES

Generalized definite languages were introduced in [GIN] and the characterization of their syntactic semigroups was found by several authors [BR-SI,PER,ZAL]. In this section we provide a brief summary of these results.

DEFINITION 1. Let  $X \subset A^+$ . Then

- (a)  $X$  is finite/cofinite iff either  $X \in F$  or  $\bar{X} \in F$ .
- (b)  $X$  is definite iff  $X = E \cup A^*F$ , for some  $E, F \in F$ .
- (c)  $X$  is reverse definite iff  $X = E \cup FA^*$ , for some  $E, F \in F$ .
- (d)  $X$  is generalized definite iff  $X = E \cup \bigcup_{i \in I} G_i A^* H_i$ , for some  $E \in F$  and  $G_i, H_i \in F$  for all  $i \in I$ , where  $I$  is a finite index set.

These languages are related to certain congruences about to be defined. Denote the length of a word  $w \in A^*$  by  $|w|$ . For  $n \geq 0$  define  $f_n(w)$  as follows: If  $|w| \leq n$  then  $f_n(w) = w$ ; otherwise  $f_n(w)$  is the prefix of length  $n$  of  $w$ . Similarly,  $t_n(w) = w$  if  $w \leq n$ , and  $t_n(w)$  is the suffix of length  $n$  of  $w$  otherwise. Note that  $f_0(w) = t_0(w) = 1$  for all  $w \in A^*$ .

DEFINITION 2. For  $w, w' \in A^+$ ,  $n \geq 0$  define:

- (a)  $w \underset{n}{\sim} w'$  iff (i)  $w = w'$  if  $|w| < n$ ;  
(ii)  $|w'| \geq n$  if  $|w| \geq n$ .
- (b)  $w \overset{n}{\leftarrow} w'$  iff  $t_n(w) = t_n(w')$ .
- (c)  $w \overset{n}{\rightarrow} w'$  iff  $f_n(w) = f_n(w')$ .
- (d)  $w \leftrightarrow_n w'$  iff  $w \overset{n}{\leftarrow} w'$  and  $w \overset{n}{\rightarrow} w'$ .

Each of these four relations is a congruence of finite index on  $A^+$ .

DEFINITION 3. Define the families  $\beta_1, \beta_{2L}, \beta_{2R}$  and  $\beta_2$  as follows:

- (a)  $\beta_1 = F \cup C$ .
- (b)  $\beta_{2L} = (F^2 \cup CF \cup C^2)B$ .
- (c)  $\beta_{2R} = (F^2 \cup FC \cup C^2)B$ .
- (d)  $\beta_2 = (F \cup C)^2 B$ .

The three definitions above are related in the following theorems. In all cases  $X \subset A^+$ , and  $S$  is its syntactic semigroup.

THEOREM  $L \wedge R$ . The following are equivalent:

- (X1)  $X$  is finite/cofinite.
- (X2)  $X$  is a union of congruence classes of  $\sim_n$  for some  $n \geq 0$ .
- (X3)  $X \in \beta_1$ .
- (S1)  $S$  is finite, and for all  $e = e^2 \in S$ ,  $eS \cup Se = e$ .  
(Every idempotent is a zero, i.e. there is only one idempotent  $e = 0$ .)

THEOREM  $L$  (respectively  $R$ ). The following are equivalent:

- (X1)  $X$  is definite (respectively reverse definite).
- (X2)  $X$  is a union of congruence classes of  $\leftarrow_n$  (respectively  $\rightarrow_n$ ) for some  $n \geq 0$ .
- (X3)  $X \in \beta_{2L}$  (respectively  $\beta_{2R}$ ).
- (S1)  $S$  is finite, and for all  $e = e^2 \in S$ ,  $Se = e$  (respectively  $eS = e$ ). (Every idempotent is a right (respectively left) zero.)

THEOREM  $L \vee R$ . The following are equivalent:

- (X1)  $X$  is generalized definite.
- (X2)  $X$  is a union of congruence classes of  $\leftrightarrow_n$  for some  $n \geq 0$ .
- (X3)  $X \in \beta_2$ .
- (S1)  $S$  is finite, and for all  $e = e^2 \in S$ ,  $eS \cap Se = eSe = e$ .  
(Every idempotent is a "middle zero".)

The proofs of these theorems can be found in [BR-SI,PER,ZAL]. Also a comprehensive discussion of these problems is given in [EIL].

#### 4. THE FAMILY $\gamma_1$ and $J$ -TRIVIAL MONOIDS

Define the shuffle operator  $[$  as follows. For  $w = a_1 \dots a_n \in A^*$ ,  $n \geq 1$ ,  $[w \stackrel{\Delta}{=} A^* a_1 A^* a_2 \dots a_n A^*$ , and  $[1 = A^*$ . Further, let  $[w = \{[w | w \in A^*\}$  and let  $\gamma_1 \stackrel{\Delta}{=} ([w)B$ . This family of languages was studied by Simon [SIM1,SIM2].

For  $n \geq 0$ ,  $w, w' \in A^*$ , define  $w \sim_n w'$  iff  $(w \in [x$  iff  $w' \in [x)$ , for all  $x \in A^*$  such that  $|x| \leq n$ . One easily verifies that  $\sim_n$  is a congruence of finite index on  $A^*$ .

A monoid  $M$  is  $J$ -trivial iff  $MmM = Mm'M$  implies  $m = m'$  for all  $m, m' \in M$ . The correspondence between  $\gamma_1$  and  $J$ -trivial monoids was established by Simon as in the theorem below, except for the observation that (M1) is equivalent to (M2), which is our contribution.

For any monoid  $M$  and  $m \in M$  define  $P_m \triangleq \{m' \mid m \in Mm'M\}$ , and  $M_m \triangleq P_m^*$ . We can view  $P_m$  as the "alphabet" of  $m$  in  $M$ , i.e. the set of all elements of  $M$  with which  $m$  "can be written". Note that  $P_m$  is "prime" in the sense of Schützenberger [SCH2], i.e. that  $m_1 m_2 \in P_m$  implies  $m_1, m_2 \in P_m$ .

Below,  $X \subset A^*$  is a language and  $M$  is its syntactic monoid.

THEOREM  $L \wedge R$ . (Simon) The following are equivalent:

- (X2)  $X$  is a union of congruence classes of  $\sim_n$ , for some  $n \geq 0$ .
- (X3)  $X \in \gamma_1$ .
- (M1)  $M$  is finite, and for all  $e = e^2 \in M$ ,  $eM_e \cup M_e e = e$ .  
(Every idempotent is a "local zero", over its "alphabet"  $P_e$ .)
- (M2)  $M$  is finite and  $J$ -trivial.

The equivalence of (M1) and (M2) will become evident in the next section.

## 5. $L$ -TRIVIAL AND $R$ -TRIVIAL MONOIDS

A monoid  $M$  is  $L$ -trivial (respectively  $R$ -trivial) iff  $Mm = Mm'$  (respectively  $mM = m'M$ ) implies  $m = m'$ , for all  $m, m' \in M$ .

THEOREM  $L$  (respectively  $R$ ). The following are equivalent:

- (M1)  $M$  is finite and for all  $e = e^2 \in M$ ,  $M_e e = e$  (respectively  $eM_e = e$ ).  
(Every idempotent is a "local right (respectively left) zero".)
- (M2)  $M$  is finite and  $L$ -trivial (respectively  $R$ -trivial).

PROOF. (M1)  $\Rightarrow$  (M2) Suppose  $Mm = Mm'$ . Then  $m = u'm'$  and  $m' = um$  for some  $u, u' \in M$ . Thus  $m = (u'u)m = (u'u)^n m$  for all  $n \geq 1$ . Since  $M$  is finite we can choose  $n$  so that  $(u'u)^n \triangleq e$  is an idempotent. Now  $m = em$  and  $m' = uem$ . Clearly,  $u \in M_e$  and by (M1)  $ue = e$ . Thus  $m' = m$

and (M1) implies (M2).

(M2) => (M1) We first show that (M2) implies (M2)':

(M2)' M is finite and  $m_1 m_2 m_3 = m_3$  implies  $m_2 m_3 = m_3$ , for all  $m_1, m_2, m_3 \in M$ .

If  $m_1 m_2 m_3 = m_3$  then  $M m_3 = M m_1 m_2 m_3 \subset M m_2 m_3$ . Since  $M m_2 m_3 \subset M m_3$  we have  $M m_2 m_3 = M m_3$ . Since M is L-trivial,  $m_2 m_3 = m_3$  and (M2) implies (M2)'.

Finally, let  $e = e^2 \in M$  and let  $m \in M_e$ . If  $m = 1$  then  $le = e$ . Otherwise,  $m = m_1 \dots m_p$ ,  $m_i \in P_e$ ,  $1 \leq i \leq p$ . Hence  $e = u_i m_i v_i$  for some  $u_i, v_i \in M$  for each  $i$ . Now  $e = ee = (u_i m_i) v_i e$ . By (M2)',  $e = v_i e$ , and now  $e = u_i m_i e$ . Again by (M2)' we conclude  $e = m_i e$ . Since this is true for all  $m_i$ ,  $1 \leq i \leq p$ , we find  $e = m_1 \dots m_p e = me$ . Hence (M2)' implies (M1).

This concludes the proof of Theorem L. Theorem R follows by left-right duality.

Since M is J-trivial iff it is L-trivial and R-trivial, the equivalence of (M1) and (M2) in Theorem  $L \wedge R$  is now obvious.

## 6. THE FAMILY K OF MONOIDS

The family K of monoids was recently studied and characterized by Schützenberger [SCH2]. For a given monoid M let  $\equiv$  be the smallest congruence such that  $m \equiv m^2$  and  $mm' \equiv m'm$  for all  $m, m' \in M$ . A monoid M is in the family  $\tilde{K}$  iff for all idempotents  $e, f \in M$ ,  $e \equiv f$  implies  $MeM = MfM$ . M is aperiodic [SCH1,EIL] iff each subgroup of M is trivial, i.e. consists of one element only. M is H-trivial iff  $mM = m'M$  and  $Mm = Mm'$  implies  $m = m'$ . It is well known that for finite M, M is aperiodic iff it is H-trivial [SCH1,EIL]. We now define the family K as follows. A monoid M is in K iff it is in  $\tilde{K}$  and is finite and aperiodic.

Following [SCH2], for  $D \subset M$  we define  $D^{-1}D = \{m \in M \mid Dm \cap D \neq \emptyset\}$ .

LEMMA 1. (Schützenberger) Let D be a non-empty subset of a monoid M. Then the following two conditions are equivalent:

- (a) D is the minimum ideal of a prime submonoid of M.
- (b) D is a J-class and a semigroup.

These conditions imply:

(c)  $P \stackrel{\Delta}{=} D^{-1}D$  is the prime submonoid whose minimum ideal is D.

THEOREM 1. (Schützenberger) Let M be a finite monoid. Then M is in  $\tilde{K}$  iff the J-class of each idempotent in M is a semigroup.

We apply these results below.

LEMMA 2. Let M be a finite monoid, let  $e = e^2 \in M$ , and let D be the J-class of e. If D is a semigroup then D is the minimum ideal of  $M_e$ .

PROOF. We will first prove that  $D^{-1}D = P_e$ . If  $m \in D^{-1}D$  then there exist  $d_m$  and  $d$  in D such that  $d_m m = d$ . Since  $e, d \in D$ , we have  $e \in MdM = Md_m m M \subset MmM$ . Hence  $m \in P_e$  and  $D^{-1}D \subset P_e$ . Conversely, let  $m \in P_e$ . Then  $e = umv = eumv$  for some  $u, v \in P_e$ . Thus  $eu, eum \in D$ . Now  $(eum) = (eu)m$  shows that  $Dm \cap D \neq \emptyset$  and  $m \in D^{-1}D$ . Thus  $P_e \subset D^{-1}D$  and our claim follows. Since D is a J-class and a semigroup, D is the minimum ideal of  $D^{-1}D$  by Lemma 1. Since  $D^{-1}D$  is a monoid, we have  $D^{-1}D = P_e = P_e^* = M_e$ .

THEOREM 1  $\vee$  R. The following are equivalent:

- (M1) M is finite and for all  $e = e^2 \in M$ ,  $eM_e \cap M_e e = eM_e e = e$ .  
(Every idempotent is a "local middle zero").
- (M2) M is in  $\tilde{K}$ .

PROOF. Let  $e = e^2 \in M$  and let D be the J-class of e. Note that  $m \in D$  implies  $M_m = M_e$ . If  $m, m' \in D$ , then  $e = u_1 m u_2$ ,  $m = v_1 e v_2$ ,  $e = u_1' m' u_2'$ ,  $m' = v_1' e v_2'$ , for some  $u_1, u_2, v_1, v_2, u_1', u_2', v_1', v_2' \in M_e$ . Now

$$\begin{aligned} e = ee &= u_1 m u_2 u_1' m' u_2' = u_1 v_1 e (v_2 u_2 u_1' v_1') e v_2' u_2' \\ &= u_1 v_1 e (v_2 v_1') e v_2' u_2' = u_1 m m' u_2', \end{aligned}$$

if (M1) holds. Hence  $e \in M m m' M$  and  $m m' \in D$ , showing that D is a semigroup. By Theorem 1, M is  $\tilde{K}$ . Suppose G is a subgroup of M with identity e. For every  $g \in G$ ,  $g g^{-1} = e$  showing that  $g \in D$ . However, each element m of D is idempotent since  $m^2 = v_1 e v_2 v_1 e v_2 = v_1 e v_2 = m$ , by (M1). Hence  $g^2 = g = g e = g g g^{-1} = e$  and M is aperiodic. Altogether (M1) implies (M2).

Conversely, let  $e = e^2 \in M$  and let  $D$  be the  $J$ -class of  $e$ . By Theorem 1,  $D$  is a semigroup. By Lemma 2,  $D$  is the minimum ideal of  $M_e$ . Thus  $D = M_e e M_e$ . Let  $m \in M_e$ ; then  $eme \in M_e e M_e = D$ . Since  $M$  is finite,  $D$  is a  $\mathcal{D}$ -class of  $M_e$  and the elements  $e$  and  $eme$  are in the same  $L$ -class and in the same  $R$ -class contained in  $D$ , i.e.  $e$  and  $eme$  are in the same  $H$ -class. Since  $M$  is aperiodic, i.e.  $H$ -trivial, we have  $e = eme$ . Hence  $e = eM_e e$ , and (M2) implies (M1).

The analogy between Section 3 and Sections 4, 5 and 6 is not quite complete, since we lack the analogous results concerning the languages corresponding to these monoids. These results can be obtained for an alphabet of two letters, as is shown next.

## 7. THE TWO-LETTER CASE

As has been mentioned before, if  $\text{card } A \leq 2$ , the family  $\gamma_1$  coincides with  $F_{\oplus} \cup C_{\oplus} = L_{\oplus} MB$ , the family of "run" languages discussed in [BRZ]. We now explore this coincidence further.

Each  $w \in A^+$  can be written in the form  $w = w_1 \dots w_p$ ,  $p \geq 1$ , where  $w_i = a_i^{n_i}$ ,  $a_i \in A$ ,  $n_i \geq 1$  for  $1 \leq i \leq p$ , and  $a_i \neq a_{i+1}$  for  $1 \leq i \leq p-1$ . We call this the run form of  $w$  and  $\|w\| \stackrel{\Delta}{=} p$  is the run length of  $w$ . Define  $\|1\| = 0$ .

For  $w \in A^*$  define  $w\alpha \stackrel{\Delta}{=} \{a \in A \mid w = xax' \text{ for some } x, x' \in A^*\}$ .

DEFINITION 4. Let  $w, w' \in A^*$  and  $n \geq 1$ . We define  $w \oplus_n w'$ :

Case 1:  $\|w\| \leq 1$ .

- (a)  $|w| < n$ . Then  $w \oplus_n w'$  iff  $w = w'$ .
- (b)  $|w| \geq n$ . Then  $w \oplus_n w'$  iff  $w\alpha = w'\alpha$  and  $|w'| \geq n$ .

Case 2:  $\|w\| > 1$ .

- (a)  $\|w\| \leq n$ . Then  $w \oplus_n w'$  iff  $\|w\| = \|w'\|$  and, if  $w = w_1 \dots w_p$ ,  $w' = w'_1 \dots w'_p$  are the run forms of  $w$  and  $w'$ , then  $w_i \oplus_n w'_i$ ,  $1 \leq i \leq p$ .
- (b)  $\|w\| > n$ . Then  $w \oplus_n w'$  iff  $\|w'\| > n$ .



One verifies that  $\oplus_n$  is a congruence relation of finite index on  $A^*$ . Note that for card  $A \leq 2$ ,  $w \oplus_n w'$  implies  $w\alpha = w'\alpha$ . This follows because  $\|w\| > 1$  implies  $w\alpha = A$ . This is obviously false for card  $A > 2$ .

For  $w \in A^+$ ,  $n \geq 1$ , define  $f_{\oplus_n}(w)$  as follows. If  $\|w\| \leq n$  then  $f_{\oplus_n}(w) = w$ . Otherwise, let  $w_1 \dots w_p$  be the run form of  $w$ ,  $p > n$ ; then  $f_{\oplus_n}(w) = w_1 \dots w_n$ . Similarly,  $t_{\oplus_n}(w) = w$  if  $\|w\| \leq n$ , and  $t_{\oplus_n}(w) = w_{p-(n-1)} \dots w_p$ , otherwise. For all  $n \geq 1$ , define  $f_{\oplus_n}(1) = t_{\oplus_n}(1) = 1$ .

DEFINITION 5. For  $w, w' \in A^*$ ,  $n \geq 1$ , define:

- (a)  $w \oplus_n w'$  as in Definition 4.
- (b)  $w \leftarrow \oplus_n w'$  iff  $t_{\oplus_n}(w) \oplus_n t_{\oplus_n}(w')$ .
- (c)  $w \oplus_n w'$  iff  $f_{\oplus_n}(w) \oplus_n f_{\oplus_n}(w')$ .
- (d)  $w \leftrightarrow \oplus_n w'$  iff  $w \leftarrow \oplus_n w'$  and  $w \oplus_n w'$ .

Again, one verifies that each of these relations is a congruence of finite index on  $A^*$ .

From here on we restrict our attention to the case card  $A \leq 2$ .

DEFINITION 6. Define the families  $\beta_{\oplus_1}$ ,  $\beta_{\oplus_{2L}}$ ,  $\beta_{\oplus_{2R}}$  and  $\beta_{\oplus_2}$ :

- (a)  $\beta_{\oplus_1} = F_{\oplus} \cup C_{\oplus}$ .
- (b)  $\beta_{\oplus_{2L}} = (F_{\oplus} \cup C_{\oplus} F_{\oplus} \cup C_{\oplus})B$ .
- (c)  $\beta_{\oplus_{2R}} = (F_{\oplus} \cup F_{\oplus} C_{\oplus} \cup C_{\oplus})B$ .
- (d)  $\beta_{\oplus_2} = (F_{\oplus} \cup C_{\oplus})^2 B$ .

We now state theorems analogous to Theorems  $L \wedge R$ ,  $L$ ,  $R$  and  $L \vee R$ , for card  $A \leq 2$ . For conciseness we only prove the most general case.

THEOREM  $(L \vee R)_2$ . For card  $A \leq 2$ , the following are equivalent:

- (X1)  $X$  is  $\oplus$ generalized definite, i.e.  $X = E \cup \bigcup_{i \in I} G_i A^* H_i$ , for some  $E, G_i, H_i \in F_{\oplus}$  and a finite index set  $I$ .

- (X2)  $X$  is a union of congruence classes of  $\leftrightarrow_n$  for some  $n \geq 1$ .
- (X3)  $X \in \beta_{\neq 2}$ .
- (M1)  $M$  is finite and for all  $e = e^2 \in M$ ,  $eM_e \cap M_e e = eM_e e = e$ .
- (M2)  $M$  is in  $K$ .

We first prove some preliminary results.

LEMMA 3. Let  $M$  in  $K$  be a monoid over  $A$  with morphism  $\mu$ . Let  $n = \text{card } M$  and let  $x \in A^*$ ,  $x = x_1 \dots x_p$ ,  $p \geq n$ , be such that  $x_1 \alpha = \dots = x_p \alpha = A$ . Then  $m \triangleq x\mu$  is an idempotent in the minimum ideal of  $M$ .

PROOF. Let  $x_0 = 1$ . The elements  $x_0\mu, (x_0x_1)\mu, \dots, (x_0 \dots x_p)\mu$  cannot all be distinct since  $p \geq \text{card } M$ . Hence there exist  $i, 0 \leq i < p$ , and  $j, i < j \leq p$  such that  $(x_0 \dots x_i)\mu = (x_0 \dots x_j)\mu$ . Let  $m_1 = (x_0 \dots x_i)\mu$ ,  $m_2 = (x_{i+1} \dots x_j)\mu$  and  $m_3 = (x_{j+1} \dots x_p)\mu$ . Let  $k$  be such that  $e \triangleq m_2^k$  is an idempotent. Then  $m_1 = m_1 m_2 = m_1 m_2^k = m_1 e$  and  $m = m_1 e m_3$ . Since  $(x_{i+1} \dots x_j)\alpha = A$ ,  $M_e = M_{m_2} = M$ , because  $M$  is generated by  $\{a\mu \mid a \in A\}$ . Now  $m^2 = m_1 e m_3 m_1 e m_3 \stackrel{k}{=} m_1 e m_3 = m$  by (M1) of Theorem  $L \vee R$ . By Theorem 1, the  $J$ -class  $D$  of  $e$  is a semigroup. By Lemma 2,  $D$  is the minimum ideal of  $M$ .

COROLLARY. Let  $M$  be as in Lemma 3 and let  $\text{card } A = 2$ . Then  $x \in A^*$ ,  $\|x\| \geq 2 \text{ card } M = 2n$  implies  $m \triangleq x\mu$  is an idempotent in the minimum ideal of  $M$ .

PROOF. We can write  $x$  in its run form  $x_1 \dots x_p$ ,  $p \geq 2n$ . Let  $w_i = x_{2i-1} x_{2i}$ ,  $1 \leq i \leq n-1$  and  $w_n = x_{2n-1} x_{2n} \dots x_p$ . Then the decomposition  $x = w_1 \dots w_n$  satisfies the conditions of Lemma 3.

PROPOSITION 1. Let  $n \geq 2$  and let  $w \in A^*$  have run length  $\|w\| < n$ . Then  $w \leftrightarrow_n w'$  implies  $w \oplus_n w'$ .

PROOF. If  $\|w\| < n$  then  $f_{\oplus n}(w) = w$ . Now  $w \leftrightarrow_n w'$  implies  $f_{\oplus n}(w') \oplus_n w$  and  $\|f_{\oplus n}(w')\| = \|w\| < n$ . It follows that  $\|w'\| = \|f_{\oplus n}(w')\|$  and  $f_{\oplus n}(w') = w'$ . Thus  $w \oplus_n w'$ .

PROPOSITION 2. Let  $w \in A^*$ ,  $w\alpha = B$ ,  $x \in B^*$ ,  $n \geq 1$ . Then  $w^n x w^n \leftrightarrow_n w^n$ .

PROOF. This is trivial if  $\|w\| = 0$ . If  $\|w\| = 1$ , then  $w = a^i$  for some  $a \in A$ ,  $i \geq 1$  and  $w^n = a^n a^m$ , where  $n+m = ni \geq n$ . The claim is easily verified. For  $\|w\| > 1$ , we have  $\|w^n\| > n$  and  $f_{\oplus n}(w^n) = f_{\oplus n}(w^n x w^n)$  for all  $x \in A^*$ . Similarly,  $t_{\oplus n}(w^n) = t_{\oplus n}(w^n x w^n)$ , and the claim follows.

PROOF OF THEOREM ( $L \vee R$ )<sub>2</sub>

(M1) iff (M2) follows by Theorem  $L \vee R$

(M2) implies (X2)

We want to show that if  $M$  is in  $K$ , then there exists an  $n \geq 1$  such that  $w \leftrightarrow_n w'$  implies  $w\mu = w'\mu$ .

(1) Let  $n = 2\text{card } M$ . For  $\|w\| < n$ ,  $w \leftrightarrow_n w'$  implies  $w \leftrightarrow_n w'$ , by Proposition 1.

(2) For  $\|w\| \leq 1$ ,  $w \leftrightarrow_n w'$  implies that either  $w = w'$  (and  $w\mu = w'\mu$ ) or  $|w| > n$ ,  $|w'| > n$  and  $w\alpha = w'\alpha = a \in A$ . Since  $M$  is aperiodic and  $n = 2\text{card } M$ , we must have  $a^n \mu = a^{n+1} \mu$ . Hence  $w \leftrightarrow_n w'$  implies  $w\mu = w'\mu$ .

(3) Now suppose  $1 < \|w\| \leq n$ . Then  $w \leftrightarrow_n w'$  iff  $w = w_1 \dots w_p$ ,  $w' = w'_1 \dots w'_p$  and  $w_i \leftrightarrow_n w'_i$ ,  $1 \leq i \leq p < n$ . For each  $i$ ,  $w_i \alpha = w'_i \alpha \in A$ . By (2)  $w_i \mu = w'_i \mu$  and  $w\mu = (w_1 \mu) \dots (w_p \mu) = (w'_1 \mu) \dots (w'_p \mu) = w'\mu$ . Again  $w \leftrightarrow_n w'$  implies  $w\mu = w'\mu$ .

Altogether we have shown that  $\|w\| < n$  and  $w \leftrightarrow_n w'$  implies  $w\mu = w'\mu$ .

(4) If  $\|w\| \geq n$ , then  $w \leftrightarrow_n w'$  implies  $f_{\oplus n}(w) \leftrightarrow_n f_{\oplus n}(w')$ . Then  $\|f_{\oplus n}(w)\| = \|f_{\oplus n}(w')\| = n$  and, by (3),  $(f_{\oplus n}(w))\mu = (f_{\oplus n}(w'))\mu$ . Let  $w\mu = m$ ,  $w'\mu = m'$ ,  $(f_{\oplus n}(w))\mu \triangleq e_1$ ,  $(t_n(w))\mu \triangleq e_2$ . By the corollary to Lemma 3,  $m$  and  $m'$  are idempotents in the minimum ideal  $D$  of  $M$  as are  $e_1$  and  $e_2$ . Further  $m = e_1 u = v e_2$  for some  $u, v \in M$  and  $m = e_1 m e_2$ . Similarly,  $m' = e_1 m' e_2$ . Thus  $mm' = (e_1 m e_2 e_1) m' e_2 = e_1 m' e_2 = m'$  by

(M1) and also  $mm' = e_1 m (e_2 e_1 m' e_2) = e_1 m e_2 = m$ . Hence  $m = m'$ .

(X2) implies (M1)

Let  $X$  be a union of congruence classes of  $\leftrightarrow_n$ , let  $M$  be the syntactic monoid of  $X$  and  $e = e^2 \in M$ . If  $e\mu^{-1} = 1$  then  $M_e = 1$  and (M1) holds. Hence assume  $M_e = B^* \mu$  for some non-empty  $B \subset A$ . Choose  $w \in A^*$  so that  $w\alpha = B$  and  $w\mu = e$ . This can always be done since  $e$  is an idempotent. By Proposition 2,  $w^n x w^n \leftrightarrow_n w^n$  for all  $n \geq 1$ ,  $x \in B^*$ . By (X2)  $w^n \mu = (w^n \mu)(x\mu)(w^n \mu)$  or  $e = e(x\mu)e$ . Since for each  $m \in M_e$  there exists  $x \in B^*$  such that  $x\mu = m$ , we have  $eme = e$  and  $eM_e e = e$  holds. Since  $\leftrightarrow_n$  is of finite index,  $M$  is finite.

We have now proved the equivalence of (X2), (M1) and (M2).

The proof of the equivalence of (X1), (X2) and (X3) is a straightforward extension of the corresponding proof in Theorem  $L \vee R$ .

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Received July 13, 1976; in  
final form October 13, 1976.