JOURNAL OF COMPUTER AND SYSTEM SCIENCES: 4, 260–280 (1970)

General Properties of Star Height of Regular Events

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Received September 15, 1969

Properties of star height of regular events are investigated. It is shown that star height is preserved under such operations as taking quotients, addition or subtraction of a finite event, removal of all words beginning with a given letter, and removal of certain subsets of smaller star height. Next it is shown that there exist events of arbitrarily large star height whose union, concatenation, and star is of star height one. Also, arbitrarily large increases in star height can be obtained by using the intersection or complement operations.

In the second part of the paper a technique for establishing the star height of regular events is developed. It is shown that for every regular event R of star height n there exists a nondeterministic state graph G whose states correspond to subsets of the set of states Q of the reduced automaton accepting R and whose cycle rank is precisely n. Unfortunately a given subset Q' of Q may have to be repeated k times in G and no bound on k is known. Thus it is still not known whether an algorithm for determining star height exists. However, it is felt that the techniques presented here provide new insight into the problem.

1. INTRODUCTION

Regular events have been studied extensively and many problems related to these events or to the corresponding finite automata have been answered. There does remain, however, a difficult open problem, namely the problem of determining the star height of a regular event. This problem is of importance because star height represents a measure of complexity of regular events and the minimum star height would be very desirable in any canonical form for regular expressions.

The problem was first studied by Eggan [1] who showed that there exist regular

* Part of this work was done while the authors were at the University of Ottawa in the Departments of Mathematics and Electrical Engineering, respectively.

This work was supported in part by the National Research Council of Canada under grants A-1617 and A-5255.

A portion of the results in this paper was presented at the Eighth Annual Symposium on Switching and Automata Theory, Austin, Texas, 1967.

events of arbitrary star height. This was done by exhibiting, for each n, events of star height n. A disadvantage of Eggan's examples is the fact that the size of the input alphabet increases with n. Dejean and Schutzenberger [10] have shown that for each n there exists an event over the two-letter alphabet that has star height n. (It is easy to show that every event over a one-letter alphabet has star height 0 or 1.)

Eggan has also introduced the notion of (cycle) rank of a digraph and has established a fundamental theorem, by which the star height of a regular event equals the smallest rank of all transition graphs recognizing the event.

Eggan's theorem was used by McNaughton in [3] for developing graph-theoretical methods for establishing the star height of regular events. It was also proved in that paper, that every regular event of star height n can be mapped by a homomorphism onto an event of star height n over the two-letter alphabet. Pathwise homomorphisms between transition graphs were introduced in [4]. Such homomorphisms were shown to be rank-nonincreasing and were utilized for establishing the existence of an algorithm for determining the star height of pure-group events.

In this paper we develop some general properties of star height. The effect of various operations on the star height of regular events is examined. Some operations, such as taking quotient or subtracting certain subsets from a regular event, are shown to preserve star height, whereas other operations, e.g., intersection or complementation, can cause arbitrary changes in star height. In the last section a technique for determining star height of regular events is presented.

2. Preliminaries

It is assumed that the reader is familiar with the basic properties of regular expressions, regular events, derivatives, and finite automata [8].

We use the following notation:

A—finite alphabet (alphabets $\{0, 1\}$ and $\{0, 1, 2\}$ are used in examples),
A^*—free monoid generated by A,
$w, u, v \in A^*$—words,
$l(w)$—length of a word,
λ—empty word,
$U, V, W \subseteq A^*$—arbitrary events,
$\varnothing$—empty event,
$U \cup V$—union,
$U \cap V$—intersection,
$U - V$—difference,
$\bar{U} = A^* - U$—complement,
$U \oplus V = (U - V) \cup (V - U)$—symmetric difference,

$UV = \{uv \mid u \in U, v \in V\}$—concatenation,
$U^* = \bigcup_{n=0}^{\infty} U^n$—"star",
$U \setminus W = \{v \mid v \in A^*$ and for some $u \in U$, $uv \in W\}$—left quotient of W by U (derivative of W with respect to U) [8].
R, P, Q—regular events,
E, F, G, H—regular expressions,
$| E |$—event denoted by regular expression E,

$$\delta(W) = \begin{cases} \lambda & \text{iff } \lambda \in W \\ \varnothing & \text{otherwise} \end{cases} \text{—for all } W \subseteq A^*.$$

$U^T = \{a_1 a_2 \cdots a_k \mid a_i \in A$ and $a_k \cdots a_2 a_1 \in U\}$—reverse (transpose) of U
$\{\lambda\}^T = \{\lambda\}$, $\varnothing^T = \varnothing$.

(*Restricted*) *regular expression* uses operators $\cup$, $\cdot$, $*$. *Extended regular expression* uses operators $\cup$, $\cdot$, $*$, $\cap$, $\overline{}$.

Unless otherwise specified, "regular expression" means restricted regular expression in the following text.

DEFINITION 2.1. A *finite incomplete automaton* (or simply *automaton*) $\mathscr{A}$ over alphabet A is a quadruple $\mathscr{A} = (Q, M, q_1, F)$, where Q is a finite set of *states*, $q_1 \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *final states* and M is the *transition function*, $M : Q \times A \to \{Q' \mid Q' \subseteq Q, \#Q' \leqslant 1\}$, where $\#S$ denotes the cardinality of the set S. If for all $q \in Q$ and for all $a \in A$, $\#M(q, a) = 1$, then $\mathscr{A}$ is a *complete automaton.* The transition function M is extended to words in the usual way. A word $w \in A^*$ is *accepted* by $\mathscr{A}$ iff $M(q_1, w) = q$[1] for some $q \in F$. The set of all words accepted by $\mathscr{A}$ is called the event accepted (or recognized) by $\mathscr{A}$. Denote by $\mathscr{A}_0(R)$ the reduced complete automaton accepting R [8].

Extend further the transition function M as follows: for any $Q' \subseteq Q$, $W \subseteq A^*$,

$$M(Q', W) = \{q \mid q = M(q', w) \text{ for some } q' \in Q', w \in W\}.$$

Define the function $M_1 : 2^{A^*} \to 2^Q$ by: $M_1(W) = M(q_1, W)$ for any event $W \subseteq A^*$.

The state graph of an automaton $\mathscr{A} = (Q, M, q_1, F)$ will be denoted by $G_{\mathscr{A}}$. The nodes of $G_{\mathscr{A}}$ are the states of $\mathscr{A}$ and for every pair $q, q' \in Q$ such that $M(q, a) = q'$ for some $a \in A$, there is in $G_{\mathscr{A}}$ a directed branch leading from q to q' and labelled by a. If $\mathscr{A}$ is a complete automaton, then $G_{\mathscr{A}}$ is called a *complete state graph.* Generally $G_{\mathscr{A}}$ is an *incomplete state graph.*

In the sequel, all state graphs and all automata dealt with will be incomplete, unless otherwise specified.

DEFINITION 2.2. Let $\mathscr{A}$ be an automaton accepting the event R and let q be a

[1] When no confusion arises, a singleton subset $\{q\}$ will be denoted simply by q.

state of $\mathscr{A}$. Define the event $A_c^{\mathscr{A}}(q)$ (or simply $A_c(q)$, when the reference to $\mathscr{A}$ is understood), called the *accepted event* of q in $\mathscr{A}$, by

$$A_c(q) = \{w \mid w \in A^* \text{ and } \varnothing \neq M(q, w) \in F\}.$$

$A_c(q)$ is the left quotient of R corresponding to state q, i.e., if w is a word such that $M_1(w) = q$, then $A_c(q) = w \setminus R$ [8].

Let $\mathscr{A} = (Q, M, q_1, F)$ and $G = G_{\mathscr{A}}$. We shall associate Q, M, q_1 and F with G rather than with $\mathscr{A}$ if convenient. For any set of nodes Q' of G, define $G - [Q']$ to be the graph resulting from G after deleting all nodes of Q' together with all transitions to and from Q'. A state graph G' is called a *subgraph* of G [3] iff $G' = G - [Q']$ for some $Q' \subseteq Q$.

Let $\mathscr{A} = (Q, M, q_1, F)$ be an incomplete automaton. Then $\mathscr{A}$ can be transformed into a complete automaton recognizing the same event by introducing a new state $q_\varnothing$ which corresponds to an empty quotient of R ("dead state") and defining for all pairs $(q, a) \in Q \times A$ for which M is not specified, $M(q, a) = q_\varnothing$, as well as $M(q_\varnothing, a) = q_\varnothing$ for all $a \in A$. Conversely, for every automaton $\mathscr{A}$ with state graph $G_{\mathscr{A}}$ and a dead state $q_\varnothing$, $G_{\mathscr{A}} - [\{q_\varnothing\}]$ is an incomplete state graph recognizing the same event.

Let $\mathscr{A} = \mathscr{A}_0(R)$ be the complete reduced automaton recognizing R. Let $G_0(R) = G_{\mathscr{A}} = G$ denote the corresponding complete reduced state graph recognizing R. Define $\tilde{G} = \tilde{G}_0(R)$ to be $G - [\{q_\varnothing\}]$ in case $\mathscr{A}$ has a dead state $q_\varnothing$, or $\tilde{G} = G$ otherwise. $\tilde{G}_0(R)$ will be called the *reduced incomplete state graph recognizing* R, and the corresponding automaton will be denoted by $\tilde{\mathscr{A}}_0(R)$.

DEFINITION 2.3. Let $\mathscr{A} = (Q, N, q_1, G)$ and $\mathscr{B} = (P, M, p_1, F)$ be two complete automata. Define their *direct product*

$$\mathscr{A} \times \mathscr{B} = (Q \times P, N \times M, (q_1, p_1), G \times F)$$

where the transition function $N \times M$ is defined by:

$$(N \times M)((q, p), a) = (N(q, a), M(p, a))$$

for all pairs $(q, p) \in Q \times P$ and $a \in A$.

3. THE STAR HEIGHT OF A REGULAR EVENT

DEFINITION 3.1. The *apparent star height* h_α of a regular expression E is defined inductively:

$$\begin{aligned} h_\alpha(a) &= 0 \quad \text{for} \quad a \in A, \qquad h_\alpha(\lambda) = h_\alpha(\varnothing) = 0, \\ h_\alpha(E_1 \cup E_2) &= \max\{h_\alpha(E_1), h_\alpha(E_2)\}, \\ h_\alpha(E_1 E_2) &= \max\{h_\alpha(E_1), h_\alpha(E_2)\}, \\ h_\alpha(E^*) &= h_\alpha(E) + 1. \end{aligned}$$

DEFINITION 3.2. The *star height* $h(R)$ of a regular event R is defined by

$$h(R) = \min\{h_\alpha(E) \mid E \text{ is a regular expression and } | E | = R\}.$$

Thus, for any regular expression E, $h_\alpha(E)$ is the maximum length of a sequence of stars in the expression E, such that each star is in the scope of the star that follows it. $h(E)$, however, indicates the star height of the event $| E |$ denoted by E as defined above.

Obviously, for any regular event R, $h(R) \geqslant 1$ iff R is infinite.

EXAMPLE 3.1. Let $E = (10^*1)^*$. Then $h_\alpha(E) = 2$ is the apparent star height of E. However, $h(E) = 1$ because $E_1 = \lambda \cup 1(0 \cup 11)^*1$ is an expression equivalent to E, i.e. $| E_1 | = | E |$.

EXAMPLE 3.2. Let $E = (0 \cup 10^*1)^*$. Here again $h_\alpha(E) = 2$. Moreover, this event has been shown to be of star height 2 by McNaughton [3], using graph-theoretical methods. We present here an outline of a relatively simple algebraic proof to the same effect.[2]

Suppose, by contradiction, that there exists a regular expression E' of apparent star height 1 such that $| E' | = | E |$. Let H_i^*, $i = 1,..., n$, $n \geqslant 1$, be the set of all star expressions appearing in E' (some of the H_i^*'s might be identical). Then the minimum roots H_i[3] of these star expressions must be finite events. Thus let

$$| H_i | = \bigcup_{j=1}^{k_i} w_{ij}, \qquad i = 1,..., n$$

where each w_{ij} is a word. It can be easily seen that each w_{ij} must contain an even number of 1's (otherwise there would be in $| E' |$ words with an odd number of 1's). Let r be the maximal length of a sequence of consecutive 0's appearing in any w_{ij}, $i = 1,..., n, j = 1,..., k_i$, and let t be the total number of letters 0, 1 appearing in E' outside the scope of any star operator. Then a word of the form $u_m = 0^m10^m1$, where $m > r$, cannot be a subword of any w_{ij}, nor of any product of words w_{ij}, i.e., u_m, $m > r$, is not a subword of any word in $| H_i^* |$, $i = 1,..., n$. It follows that a word $(u_m)^k$, where $m > r$ and $k > n + t$ cannot be denoted by the regular expression E'. But this contradicts the fact, that $(u_m)^k \in | E' |$ for all integers m, k.

DEFINITION 3.3. A *section* S of a state graph G is any strongly connected (s.c.) subgraph of G which is not propertly contained in any other s.c. subgraph of G.

[2] Unfortunately, such algebraic proofs not using automata can be obtained only for some special cases ([1], [10]) and do not seem to be extensible to general families of events.

[3] For any regular events R and H, H is called a *root* of R iff $R = H^*$.

DEFINITION 3.4. *The (cycle) rank of a state graph (after Eggan* [1]):

(a) Let G be a s.c. state graph. The rank $r(G)$ of G is defined inductively as follows:

1. $r(G) = 1$ iff there exists a state q of G such that $G - [\{q\}]$ has no sections.
2. $r(G) = k$ iff $r(G)$ is not less than k and for some state q of G all sections of $G - [\{q\}]$ have rank at most $k - 1$.

(b) Let G be any state graph. Define $r(G) = 0$ iff G has no sections, and

$$r(G) = \max\{r(G') \mid G' \text{ a section of } G\}$$

otherwise (for a very original and clear explanation of this notion, see [3]).

EGGAN'S THEOREM [1]: *The star height of a regular event R does not exceed the rank of any state graph recognizing R.*[4]

We now state without proof a result which will be necessary for Sections 4, 5.

DEFINITION 3.5. Let $\mathscr{A} = (Q, M, q_1, F)$ be an automaton. $\mathscr{A}$ is called *reset-free* iff for every $a \in A$, $q, q' \in Q$, $M(q, a) = M(q', a)$ implies either $q = q'$ or $M(q, a) = \varnothing$. A state graph is reset-free iff its corresponding automaton is reset-free.

The following theorem was first proved by McNaughton in [4]. A generalized version of this theorem is proved in [12], using a totally different approach.

THEOREM 3.1. *If the reduced state graph $\tilde{G}_0(R)$ of an event R is reset-free and has exactly one final state, then $h(R) = r(\tilde{G}_0(R))$.*

4. STAR-HEIGHT PRESERVING OPERATIONS

In this section as well as the next one, the effect of various operations on the star height of regular events is examined.

We first show that star height is preserved by quotients.

THEOREM 4.1. *Let R be a regular event and let $w \in A^*$. Then $h(w \setminus R) \leqslant h(R)$. Furthermore, for all integers $n \geqslant 0$, there exists a word w with $l(w) = n$ such that $h(w \setminus R) = h(R)$.*

Proof. We shall first prove that for every regular expression E such that $h_\alpha(E) = h(E)$ and for every $a \in A$, $h(a \setminus E) \leqslant h(E)$. The proof will proceed by induction on the number m of regular operators ($\cup$, $\cdot$, $*$) in E.

[4] Eggan's Theorem refers also to nondeterministic state graphs and transition graphs (see Section 6).

Basis, $m = 0$. E must be one of $\varnothing, \lambda, a_i \in A$. Clearly $h(E)$ is 0 and one easily verifies that $h(a \setminus E) = 0$ for all $a \in A$.

Induction step, $m > 0$. Assume that for all regular expressions E with $h_\alpha(E) = h(E)$ and with at most m operators, $h(a \setminus E) \leqslant h(E)$ for all $a \in A$. Let E have $m + 1$ operators. We have three cases:

(1) $E = G \cup H$. Clearly $h(E) = \max\{h(G), h(H)\}$. Also, for $a \in A$, $a \setminus E = a \setminus G \cup a \setminus H$. Thus $h(a \setminus E) \leqslant \max\{h(a \setminus G), h(a \setminus H)\}$. However, by induction assumption, $h(a \setminus G) \leqslant h(G)$ and $h(a \setminus H) \leqslant h(H)$ because G and H have at most m operators. Hence

$$h(a \setminus E) \leqslant \max\{h(G), h(H)\} = h(E).$$

(2) $E = GH$. Then again $h(E) = \max\{h(G), h(H)\}$. Also

$$a \setminus E = (a \setminus G)\, H \cup \delta(G)(a \setminus H).$$

$$h(a \setminus E) \leqslant \max\{h(a \setminus G), h(a \setminus H), h(H)\} \leqslant \max\{h(G), h(H)\} = h(E),$$

where the last inequality follows from the induction assumption.

(3) $E = G^*$. Then $a \setminus E = (a \setminus G)(G)^*$ and $h(a \setminus E) \leqslant \max\{h(a \setminus G), h(G^*)\} \leqslant \max\{h(G), h(G^*)\} = h(E)$.

So far we have shown that $h(a \setminus E) \leqslant h(E)$ for every regular expression E with $h_\alpha(E) = h(E)$ and every $a \in A$. But since every regular event R can be denoted by a regular expression E such that $h_\alpha(E) = h(R)$ we deduce that $h(a \setminus R) \leqslant h(R)$ for all regular events R. Also since $\lambda \setminus R = R$, we have that $h(w \setminus R) \leqslant h(R)$ holds for all words w of length 0 or 1. Now assume this condition holds for $l(w) \leqslant k \geqslant 1$. Let w be of length $k + 1$. Then $w = w_1 a$ where $l(w_1) = k$ and $a \in A$. But $w_1 a \setminus R = a \setminus (w_1 \setminus R)$ and hence $h(w \setminus R) = h(w_1 a \setminus R) \leqslant h(w_1 \setminus R) \leqslant h(R)$. It follows that $h(w \setminus R) \leqslant h(R)$ for all words w.

Now suppose $A = \{a_1, a_2, ..., a_k\}$ and for all $a_i \in A$, $h(a_i \setminus R) < h(R)$. Since we have [8]: $R = \bigcup_{i=1}^{k} a_i(a_i \setminus R) \cup \delta(R)$, we also have $h(R) \leqslant \max\{h(a_i \setminus R) \mid a_i \in A\} \leqslant h(R) - 1$. This is a contradiction. Hence, for at least one $a_i \in A$ we have $h(a_i \setminus R) = h(R)$. Apply the same argument to $(a_i \setminus R)$: There exists an $a_j \in A$ such that $h(a_j \setminus (a_i \setminus R)) = h(a_i \setminus R)$. Thus there exist a_i, a_j such that $h(a_i a_j \setminus R) = h(R)$. By induction on the length of w it follows that for all $n \geqslant 0$, there exists a $w \in A^*$ with $l(w) = n$ such that $h(w \setminus R) = h(R)$.

This result leads to a number of corollaries.

COROLLARY 4.2. *Let R be a regular event and let $\mathscr{A} = \mathscr{A}_0(R)$. Then if $\mathscr{A}$ is strongly connected, $h(w \setminus R) = h(R)$ for all $w \in A^*$.*

Proof. Assume there exists a $w \in A^*$ such that $h(w \setminus R) < h(R)$. Let q_w be the state corresponding to $w \setminus R$ (i.e., $q_w = M_1(w)$). Since $\mathscr{A}$ is s.c., there exists a $u \in A^*$ such that $M(q_w, u) = q_1$. In other words, $u \setminus (w \setminus R) = wu \setminus R = R$. But now $h(R) = h(u \setminus (w \setminus R)) \leqslant h(w \setminus R) < h(R)$ which is a contradiction.

COROLLARY 4.3. *Let R be a regular event and W any event. Then* $h(W \setminus R) \leqslant h(R)$.

Proof. Since $W \setminus R = \{u \mid \exists w \in W \text{ such that } wu \in R\}$, we have $W \setminus R = \bigcup_{w \in W} (w \setminus R)$. However R has only a finite number of distinct quotients $w_1 \setminus R$, $w_2 \setminus R, \ldots, w_n \setminus R$ [8]. Hence $W \setminus R$ is a finite union of some or all of the $w_i \setminus R$ and

$$h(W \setminus R) \leqslant \max\{h(w_i \setminus R),\ i = 1, \ldots, n\} \leqslant h(R).$$

COROLLARY 4.4. *Let* $R, T \subseteq A^*$ *be regular and let* $h(T) = 0$. *Then* $h(R) = h(R \cup T) = h(R - T) = h(R \oplus T)$.

Proof. Since $h(T) = 0$, T is finite. Let the longest word in T have length n. Then for $l(w) = n + 1$, $w \setminus T = \varnothing$. Thus $w \setminus (R \cup T) = w \setminus R = w \setminus (R - T)$. Hence both $R \cup T$ and $R - T$ have left quotients identical to left quotients of R for all w, $l(w) = n + 1$. Among these there exists at least one w_0 such that $h(w_0 \setminus R) = h(R)$. Since $h(R \cup T) \leqslant h(R)$ and star height cannot be increased by taking quotients, we have $h(R \cup T) = h(R)$. Similarly, $h(R - T) \geqslant h(w_0 \setminus (R - T)) = h(w_0 \setminus R) = h(R)$. But $h(R - T)$ cannot exceed $h(R)$ for we can write $R - T = S \cup (\bigcup_{l(w)=n+1} w(w \setminus R))$ where $S = \{u \mid u \in R - T \text{ and } l(u) \leqslant n\}$. Hence $h(R - T) = h(R)$. Also $h(R \oplus T) = h((R - T) \cup (T - R)) = \max\{h(R - T), h(T - R)\} = h(R - T) = h(R)$.

Thus the star height is not affected by addition or subtraction of finite events. At this point the question naturally arises, whether the star height could be affected by multiplying, or dividing an event by a finite event; i.e., is $h(TR) = h(R)$ for any event R and finite event T?

Clearly $h(TR) \leqslant h(R)$ always holds. Is it possible, however, to have a strict inequality $h(TR) < h(R)$? The following example, provided to the authors by R. McNaughton [11], shows that this is possible.

EXAMPLE 4.1. Let $A = \{a_1, a_2, a_3\}$ and let R be the set of all words over A having an even number of at least two of the three letters. Thus, if

$$R_i = ((A - \{a_i\})^* a_i(A - \{a_i\})^* a_i)^* (A - \{a_i\})^*, \qquad i = 1, 2, 3,$$

then $R = (R_1 \cap R_2) \cup (R_1 \cap R_3) \cup (R_2 \cap R_3)$. Now it can be verified, using the methods introduced in [4], that $h(R) = 3$. Let $T = \lambda \cup a_1 \cup a_2 \cup a_3$. One can also verify that $TR = \lambda \cup a_1(R_2 \cup R_3) \cup a_2(R_1 \cup R_3) \cup a_3(R_1 \cup R_2)$. Hence $h(TR) \leqslant 2$.

With a little more effort it can be shown that $h(TR) = 2$. Similarly, $h(RT) = 2$.

The following corollary shows that if the reduced automaton $\mathscr{A}_0(R)$ is strongly connected, all words beginning with a letter 'a_i' can be removed from R (provided the alphabet A has at least two letters) without affecting its star height.

COROLLARY 4.5. *Let R be a regular event and $\mathscr{A} = \mathscr{A}_0(R)$. If $\mathscr{A}$ is strongly connected and if the input alphabet A has at least two letters, then for any $a_i \in A$,*

$$h(R - a_i A^*) = h(R).$$

Proof. Let $A = \{a_1, a_2, ..., a_k\}$, and denote $a_i \diagdown R$ by R_i. Then we have [8]: $R = a_1R_1 \cup a_2R_2 \cup \cdots \cup a_kR_k \cup \delta(R)$, where clearly all the terms are disjoint. Subtracting a_iR_i $(1 \leqslant i \leqslant k)$ from both sides we get:

$$R - a_iR_i = a_1R_1 \cup \cdots \cup a_{i-1}R_{i-1} \cup a_{i+1}R_{i+1} \cup \cdots \cup a_kR_k .$$

Hence $h(R - a_iR_i) \leqslant \max_{1 \leqslant j \leqslant k}\{h(R_j)\} = h(R)$.

Now clearly $a_j \diagdown (R - a_iR_i) = R_j$ for all $1 \leqslant j \leqslant k$, $j \neq i$. Also, since $\mathscr{A}$ is s.c., $h(R_j) = h(R)$. By Theorem 4.1 we have $h(R_j) = h(a_j \diagdown (R - a_iR_i)) \leqslant h(R - a_iR_i)$ and hence $h(R) \leqslant h(R - a_iR_i)$. It follows that $h(R) = h(R - a_iR_i) = h(R - a_iA^*)$ for all input letters a_i.

In Corollary 4.4 it was shown that removing from R an event T of star height 0 does not affect star height. The question arises, whether the star height of R can be affected by subtracting from it any subevent $T \subsetneq R$ of star height $h(T) < h(R)$.

PROPOSITION 4.6. *Let R and T be any regular events such that $T \subsetneq R$ and $h(T) < h(R)$. Then $h(R - T) \geqslant h(R)$.*

Proof. If $h(R - T) < h(R)$, then $R = (R - T) \cup T$ would imply

$$h(R) \leqslant \max\{h(R - T), h(R)\} < h(R).$$

Thus the star height of R cannot be reduced by subtracting from it a subset T of smaller star height. However, as was pointed out by R. McNaughton [11], it is possible to increase $h(R)$ by subtracting from R a subset T with $h(T) < h(R)$.

EXAMPLE 4.2 [11]. Let $A = \{a_1, a_2, a_3, a_4\}$, let R_1 be the set of all words over A having an even number of a_2's and an even number of a_3's, and let T be the set of all words over A having an odd number of a_1's. Let $R = R_1 \cup T$. Note that $R - T$ is the set of all words over A having an even number of a_1's, an even number of a_2's and an even number of a_3's. Now, using Theorem 3.1, one can show that $h(R_1) = 3$; $h(T) = 2$ and $h(R - T) = 4$. Thus $h(R) = h(R_1 \cup T) \leqslant 3 < h(R - T)$ (in fact, by employing McNaughton's star height algorithm for pure-group events [4], one can show that $h(R) = 3$).

The following theorem shows a special case, when the star height of R remains unchanged after removing from it certain subsets T of smaller star height.

THEOREM 4.7. *Let R be a regular event of star height $k > 1$ such that for any two words $x, y \in A^*$, either $x \setminus R = y \setminus R$ or $(x \setminus R) \cap (y \setminus R) = \varnothing$. Then for every subset $T \subsetneq R$ such that $h(T) < k$ and $r(G_0(T)) \leqslant k$, $h(R - T) = k$.*

Proof. By Proposition 4.6, $h(R - T) \geqslant h(R)$. Consider the reduced automaton $\mathscr{A} = \mathscr{A}_0(R) = (Q, M, q_1, F)$. By Theorem 5.1 in [12], $r(\mathscr{A}) = r(\tilde{\mathscr{A}}_0(R)) = k$. Let $\mathscr{A}' = \mathscr{A}_0(T) = (P, N, p_1, G)$ be the reduced complete automaton recognizing T, and let $\mathscr{B} = \mathscr{A} \times \mathscr{A}' = (Q \times P, M \times N, (q_1, p_1), F \times G)$. Let $\mathscr{B}'$ be the automaton obtained from $\mathscr{B}$ by changing the set of final states from $F \times G$ to $F \times (P - G)$, and then removing all inaccessible states.[5] Let the set of states of $\mathscr{B}'$ be S.

It can be easily verified that the automaton $\mathscr{B}'$ thus obtained accepts the event $R - T$. We claim that $r(\mathscr{B}') \leqslant k$. To see this, let $(q, p) \in S$ be any state of $\mathscr{B}'$. Then there exists a word w such that $(M \times N)((q_1, p_1), w) = (M(q_1, w), N(p_1, w)) = (q, p)$. Now since $T \subsetneq R$, $wx \in T$ implies $wx \in R$ for any word x. Hence all words x taking p to a final state in $\mathscr{A}'$, also take q to a final state in $\mathscr{A}$, i.e., $A_c^{\mathscr{A}'}(p) \subseteq A_c^{\mathscr{A}}(q)$. But since for every two states q', q'' in Q, $A_c^{\mathscr{A}}(q') \cap A_c^{\mathscr{A}}(q'') = \varnothing$, it follows that for every state p in $\mathscr{A}'$ such that $A_c^{\mathscr{A}'}(p) \neq \varnothing$, there exists at most one state $q \in Q$ such that $(q, p) \in S$. The only state p in P which may appear in more than one pair (q, p) of S is the dead state $p_\varnothing$, if such a state exists in $\mathscr{A}'$. Thus decompose S into two disjoint subsets: $S = S' \cup S''$, where S' is the set of all pairs $(q, p) \in S$ such that $p \neq p_\varnothing$, and S'' is the set of all pairs of the form $(q, p_\varnothing)$ appearing in S. We claim:

(1) There does not exist in $\mathscr{B}'$ any transition from a state of S'' to a state of S'.

(2) The subgraph $G_{\mathscr{B}'} - [S'']$ of $G_{\mathscr{B}'}$ is isomorphic with some subgraph of $G_{\mathscr{A}'}$.

(3) The subgraph $G_{\mathscr{B}'} - [S']$ of $G_{\mathscr{B}'}$ is isomorphic with some subgraph of $G_{\mathscr{A}}$.

The first assertion follows from the fact, that for any word x, $N(p_\varnothing, x) = p_\varnothing$. As for (2), let $(q, p) \in S'$. Then for any $a \in A$, $(M \times N)((q, p), a) = (M(q, a), N(p, a))$. If $N(p, a) = p_\varnothing$, then this transition is not contained in $G_{\mathscr{B}'} - [S'']$. Thus suppose $N(p, a) \neq p_\varnothing$. Define a mapping $\varphi : S' \to P - \{p_\varnothing\}$ by $\varphi(q, p) = p$ for all $(q, p) \in S'$. Then by the above remark, φ is a 1-to-1 mapping. Moreover, if for some $(q, p) \in S'$ and $a \in A$, $(M \times N)((q, p), a) = (q', p')$, then by definition, $p' = N(p, a)$. Hence φ is an isomorphism of $G_{\mathscr{B}'} - [S'']$ onto some subgraph of $G_{\mathscr{A}'}$ as required.

To show (3), let $Q' = \{q \in Q \mid (q, p_\varnothing) \in S''\}$. Define a mapping $\psi : S'' \to Q'$ by $\psi(q, p_\varnothing) = q$ for all $(q, p_\varnothing) \in S''$. Obviously ψ is an isomorphism of $G_{\mathscr{B}'} - [S']$ onto the subgraph $G_{\mathscr{A}} - [Q - Q']$, which proves (3).

[5] A state is inaccessible iff it cannot be reached from the initial state by any input sequence.

Now by (1), the set of states of any section in $C_{\mathscr{B}'}$ must be contained in either S' or S''. Thus:

$$r(G_{\mathscr{B}'}) = \max\{r(G_{\mathscr{B}'} - [S']), r(G_{\mathscr{B}'} - [S''])\} \leqslant \max\{r(G_{\mathscr{A}}), r(G_{\mathscr{A}'})\} \leqslant k.$$

Hence by Eggan's Theorem $h(R - T) \leqslant k$. This completes the proof.

Finally, we note that star height is preserved under reversal. Though this seems like a trivial observation there are nontrivial applications of this result.

EXAMPLE 4.3. Let R be the event accepted by the reduced automaton in Fig. 1. It is by no means easy to show from this state graph that $h(R) = 1$. Yet by looking at the reduced state graph for R^T (Fig. 2), one immediately sees how a regular expression of apparent star height 1 can be written for R^T.

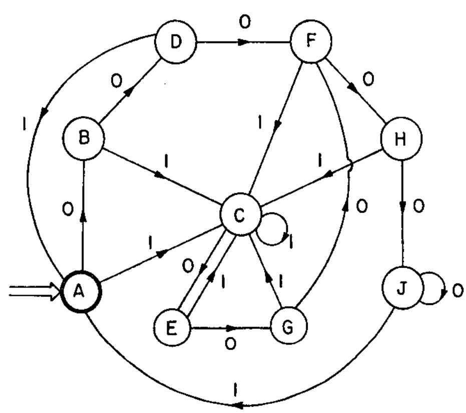

FIG. 1. The reduced state graph of R.

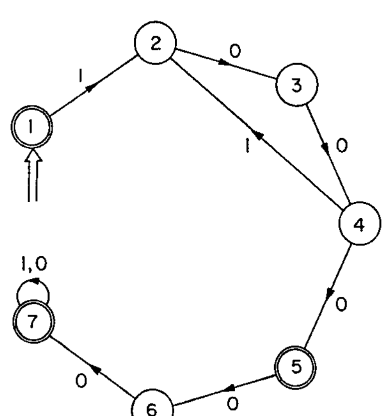

FIG. 2. The reduced state graph of R^T.

5. Operations Which do not Preserve Star Height

Next we examine the following problem. Let T_1 and T_2 be regular events and let R be an event obtained from T_1, T_2 by application of the operators $\cup$, $\cap$, $\overline{}$, $\cdot$ or $*$. How is the star height of R related to those of T_1 and T_2?—The answer is rather disappointing because in most cases very little connection appears to exist between $h(T_1)$, $h(T_2)$ and $h(R)$. We now proceed to examine specific cases.

Proposition 5.1. *For any regular events T_1, T_2, $h(T_1 \cup T_2) \leqslant \max\{h(T_1), h(T_2)\}$. Moreover, for every integer $n \geqslant 0$ there exist events T_1 and T_2 such that $h(T_1) = n$ and $h(T_1 \cup T_2) = 1$.*

Proof. The first assertion is trivial. As for the second, simply take any event T_1 of star height n and $T_2 = \overline{T_1}$. Then $h(T_1 \cup T_2) = h(A^*) = 1$.

Proposition 5.2. *For any regular events T_1, T_2, $h(T_1T_2) \leqslant \max\{h(T_1), h(T_2)\}$, and for any integer $n \geqslant 0$ there exist T_1, T_2 such that $h(T_1) = n$ and $h(T_1T_2) = 1$.*

Proof. Let T_1 be any event of star height n containing the empty word λ and let $T_2 = A^*$. Then $h(T_1T_2) = 1$.

Remark. The following stronger version of Propositions 5.1, 5.2 can be proved: for any $n \geqslant 0$ there exist events S_n and T_n such that $h(T_n) = h(S_n) = n$ and $h(S_n \cup T_n) = h(S_nT_n) = 1$.

To see this, let S_n and T_n be defined in the following way: let $A = \{a_1, a_2, ..., a_n\}$ be the alphabet and let N denote the set of integers from 1 to n. Let R_i, $1 \leqslant i \leqslant n$, be the event consisting of all words over A with an even number of a_i's (see Examples 3.2 and 4.1) and define

$$S_n = \bigcup_{\substack{N' \subseteq N, \# N' = n-2j \\ j=0,1,\ldots,[n/2]}} \left[\left(\bigcap_{i \in N'} R_i \right) \cap \left(\bigcap_{i \in N-N} \overline{R_i} \right) \right].$$

Define $T_n = \overline{S_n} \cup \{\lambda\}$. One can easily see that $\mathscr{A}_0(S_n)$ can be represented as a direct product of n 2-state automata $\mathscr{A}_1, ..., \mathscr{A}_n$, where each $\mathscr{A}_i$ is a permutation automaton[6] recognizing the event R_i or $\overline{R_i}$. Thus $\mathscr{A}_0(S_n)$ and $\mathscr{A}_0(T_n - \{\lambda\})$ are both permutation automata and one can use McNaughton's algorithm [4] to show that $h(S_n) = h(T_n - \{\lambda\}) = h(T_n) = n$. However, $S_n \cup T_n = S_nT_n = A^*$ is of star height 1.

Proposition 5.3. *For any regular event T, $h(T^*) \leqslant h(T) + 1$ and for any integer $n \geqslant 0$ there exists an event T of star height n such that $h(T^*) = 1$.*

[6] A *permutation* (or *pure-group*) automaton is any reset-free complete automaton.

Proof. Let T' be any event over alphabet A of star height n and define $T = T' \cup \{a \mid a \in A\}$. Then by Corollary 4.4 $h(T') = h(T) = n$ and $h(T^*) = h(A^*) = 1$.

PROPOSITION 5.4. *For every integer $m \geqslant 0$, there exists a regular event R such that $h(R) - h(\bar{R}) \geqslant m$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be the alphabet, where $n = m + 2 \geqslant 2$, and let $\mathscr{A}_i = (Q_i, M_i, q_1{}^i, \{q_1{}^i\})$, $i = 1, 2, \ldots, n$, be the reduced deterministic automaton recognizing the event R_i, consisting of all words over A with an even number of occurrences of the letter a_i. Let $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2 \times \cdots \times \mathscr{A}_n = (Q, M, q_1, F)$, where $Q = Q_1 \times \cdots \times Q_n$, $q_1 = (q_1{}^1, \ldots, q_1{}^n)$ and $F = \{q_1\}$. Since each $\mathscr{A}_i$ is a permutation automaton, so is $\mathscr{A}$ and by Theorem 3.1, $h(R) = r(G_{\mathscr{A}})$, where R is the event recognized by $\mathscr{A}$. If we show that $r(G_{\mathscr{A}}) \geqslant n$, then we have $h(R) \geqslant n$. Furthermore, since $R = R_1 \cap R_2 \cap \cdots \cap R_n$, by De Morgan's law, $\bar{R} = \bar{R}_1 \cup \bar{R}_2 \cup \cdots \cup \bar{R}_n$, and since each $\bar{R}_i$ is of star height 2 (see Example 3.2), we get $h(\bar{R}) \leqslant 2$. Hence $h(R) - h(\bar{R}) \geqslant n - 2 = m$ as desired.

Thus it remains to show that $r(G_{\mathscr{A}}) \geqslant n$. We shall prove this by induction on n. For $n = 2$, $\mathscr{A}_1 \times \mathscr{A}_2$ is shown in Fig. 3 and apparently has rank 2. Now suppose the

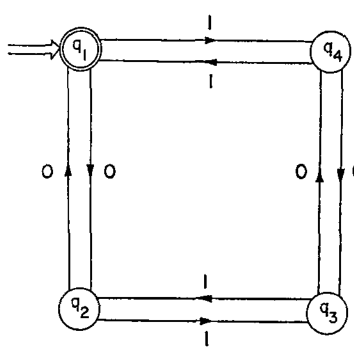

FIG. 3. The automaton $\mathscr{A}_1 \times \mathscr{A}_2$.

assertion holds for $n - 1$ and let $\mathscr{A} = \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$. Then for any node $q = (q^1, \ldots, q^n)$ of $G_{\mathscr{A}}$, the graph $G_{\mathscr{A}} - [q]$ certainly contains the subgraph $G_{\mathscr{A}} - [N']$, where

$$N' = \{(p_1, \ldots, p_n) \mid p_i \in Q_i, i = 1, \ldots, n, \text{ and } p_n = q^n\}.$$

But since Q_n (as well as the other Q_i's) consists of only two states, this graph $G_{\mathscr{A}} - [N']$ can be easily seen to be isomorphic with $G_{\mathscr{A}_1 \times \cdots \times \mathscr{A}_{n-1}}$. Hence by the induction hypothesis $r(G_{\mathscr{A}} - [q]) \geqslant r(G_{\mathscr{A}} - [N']) \geqslant n - 1$ for any state q of $\mathscr{A}$, and therefore $r(G_{\mathscr{A}}) \geqslant n$. This completes the proof.

COROLLARY 5.5. *For every integer $n \geqslant 0$, there exist regular events $R_1, R_2, \ldots, R_n$ such that*

$$h(R_1 \cap R_2 \cap \cdots \cap R_n) - \max_{1 \leqslant i \leqslant n} \{h(R_i)\} \geqslant n - 2.$$

COROLLARY 5.6. *For any integer $k > 0$ there exists a regular event R such that*

$$r(G_0(R)) - h(R) > k.$$

Proof. The event $\bar{R} = \bar{R}_1 \cup \bar{R}_2 \cup \cdots \cup \bar{R}_n$ defined in the proof of Proposition 5.4 can be easily shown to be accepted by the automaton $\bar{\mathscr{A}}$ obtained from $\mathscr{A} = \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$ by interchanging the set F of final states with the set $Q - F$. Obviously $\bar{\mathscr{A}}$ is also reduced and the result follows.

The last corollary shows that the rank of the reduced state graph $G_0(R)$ of R may differ from the star height of R by any arbitrary number. Thus, in the general case, $r(G_0(R))$ provides little information about $h(R)$. However, as is shown in [12], for certain families of regular events, $r(\tilde{G}_0(R))$ equals the star height of R. Moreover, in many other cases, lower bounds to the star height of R can be obtained by observing certain properties of some subgraphs of $G_0(R)$ [7].

6. A Technique for Determining Star Height

DEFINITION 6.1. A nondeterministic (finite) automaton $\mathscr{A}$ over alphabet A is a quadruple $\mathscr{A} = (Q, M, Q_0, F)$, where Q is a finite set of states, $F \subseteq Q$ is the set of final states, $Q_0 \subseteq Q$ is the set of initial states and M is a function: $M : Q \times A \to 2^Q$.

The transition function is extended to $2^Q \times A$: If $Q' \subseteq Q$ and $a \in A$, then $M(Q', a) = \bigcup_{q \in Q'} M(q, a)$. Extend M to $2^Q \times A^*$ by:

$$M(Q', \lambda) = Q'$$
$$M(Q', xa) = M(M(Q', x), a)$$

for any $x \in A^*$, $a \in A$ and $Q' \subseteq Q$.

The event R *recognized* (or *accepted*) by $\mathscr{A}$ is defined by:

$$R = \{w \in A^* \mid M(Q_0, w) \cap F \neq \varnothing\}.$$

For any state $q \in Q$, define the *preceding event* $P_r^{\mathscr{A}}(q)$ of q in $\mathscr{A}$ by:

$$P_r^{\mathscr{A}}(q) = \{w \in A^* \mid q \in M(Q_0, w)\}.$$

Two nondeterministic automata are *equivalent* iff they recognize the same event.

The state graph of a nondeterministic automaton is called a nondeterministic state graph. The *(cycle) rank* $r(\mathscr{A})$ of a nondeterministic automaton is the rank of its state graph.

We now recall a modified version of Eggan's Star Height Theorem (for a proof of this version, the reader is referred to [5]).

EGGAN'S STAR HEIGHT THEOREM (MODIFIED VERSION): *For any regular event R,*

$$h(R) = \min\{r(\mathscr{A}) \mid \mathscr{A} \text{ a nondeterministic automaton recognizing } R\}.$$

DEFINITION 6.2. Let $\mathscr{A} = (Q, M, Q_0, F)$ and $\mathscr{A}' = (Q', M', Q_0', F')$ be two nondeterministic automata. $\mathscr{A}'$ is said to be a *partial automaton* of $\mathscr{A}$ iff $Q' \subseteq Q$, $Q_0' = Q_0 \cap Q'$, $F' = F \cap Q'$ and for all $a \in A$, $q \in Q'$, $M'(q, a) \subseteq M(q, a)$.

Given a reduced automaton $\tilde{\mathscr{A}} = \tilde{\mathscr{A}}_0(R) = (Q, M, q_1, F)$ recognizing R, we now construct, for any integer $k > 0$, a new automaton $\mathscr{A}_k(R)$, whose states are represented by nonempty subsets of Q. $\mathscr{A}_k(R)$ has exactly k duplicates of each nonempty subset Q' of Q, each denoted by $Q'_{(i)}$, $1 \leqslant i \leqslant k$. Moreover, there is a transition in $\mathscr{A}_k(R)$ from any duplicate of a subset Q' by input $a \in A$ to any duplicate of a subset $Q'' \subseteq Q$, provided $M(Q', a) \subseteq Q''$. The initial states in $\mathscr{A}_k(R)$ are all duplicates of subsets Q' containing q_1, and the final states are all duplicates of subsets Q'' contained in F. Formally, we have:

DEFINITION 6.3. Let $\tilde{\mathscr{A}} = \tilde{\mathscr{A}}_0(R) = (Q, M, q_1, F)$. For any integer $k > 0$, define a nondeterministic automaton, called *the subset automaton* $\mathscr{A}_k(R)$ *of order k of R* as follows:

$$\mathscr{A}_k(R) = (Q^k, M_k, Q_0^k, F^k)$$

where

$$Q^k = \{Q'_{(i)} \mid \varnothing \neq Q' \subseteq Q_{(i)}, 1 \leqslant i \leqslant k\},$$

for any $\varnothing \neq Q'$, $Q'' \subseteq Q$, $a \in A$ and for any $1 \leqslant i, j \leqslant k$, $Q''_{(j)} \in M_k(Q'_{(i)}, a)$ iff $M(Q', a) \subseteq Q''$, and

$$Q_0^k = \{Q'_{(i)} \mid 1 \leqslant i \leqslant k, q_1 \in Q_i^{(\prime)}\}$$

$$F^k = \{Q'_{(i)} \mid 1 \leqslant i \leqslant k, Q'_{(i)} \subseteq F\}.$$

The following proposition can be easily verified.

PROPOSITION 6.1. *For any $k = 1, 2, \ldots$, $\mathscr{A}_k(R)$ recognizes the event R. Moreover, any partial automaton of $\mathscr{A}_k(R)$ recognizes a subset of R.*

EXAMPLE 6.1. Let $\mathscr{A} = \mathscr{A}_0(R_1)$ be the automaton in Fig. 4. Note that initial states are indicated by arrows, and final states by 1's in the rightmost column. The subset automaton of order 1, $\mathscr{A}_1(R_1)$, of R_1, is shown in Fig. 5.

	0	1	2	
⟹ p	p	r	q	1
q	r	q	p	0
r	r	r	r	1

FIG. 4. The automaton $\mathscr{A}_0(R_1)$.

	0	1	2	
⟹ $\{p\} \equiv a$	a,d,e,g	c,e,f,g	b,d,f,g	1
$\{q\} \equiv b$	c,e,f,g	b,d,f,g	a,d,e,g	0
$\{r\} \equiv c$	c,e,f,g	c,e,f,g	c,e,f,g	1
⟹ $\{p,q\} \equiv d$	e,g	f,g	d,g	0
⟹ $\{p,r\} \equiv e$	e,g	c,e,f,g	f,g	1
$\{q,r\} \equiv f$	c,e,f,g	f,g	e,g	0
⟹ $\{p,q,r\} \equiv g$	e,g	f,g	g	0

FIG. 5. The subset automaton $\mathscr{A}_1(R_1)$.

DEFINITION 6.4. Let $\mathscr{A} = \tilde{\mathscr{A}}_0(R) = (Q, M, q_1, F)$ and let $\mathscr{A}' = (Q', M', Q_0', F')$ be any arbitrary nondeterministic automaton (not necessarily equivalent to $\mathscr{A}$). Define a function $f_R : Q' \to 2^Q$ as follows: for each state q' of $\mathscr{A}'$,

$$f_R(q') = M(q_1, P_r^{\mathscr{A}'}(q')).$$

Thus $f_R(q')$ is the set of all states of $\mathscr{A}$ reachable from q_1 by words which take some initial state of $\mathscr{A}'$ to state q'. Clearly, for all initial states $q_0' \in Q_0'$ of $\mathscr{A}'$, $q_1 \in f_R(q_0')$.

EXAMPLE 6.2. Let $\tilde{\mathscr{A}}_0(R)$ and $\mathscr{A}'$ be the automata shown in Figs. 6 and 7, respectively. For each state q' of $\mathscr{A}'$, the subset $f_R(q')$ is indicated besides the circle representing q'.

LEMMA 6.2. *Let $\mathscr{A}$ and $\mathscr{A}'$ be defined as in Definition 6.4. If for some $q', q'' \in Q'$, $q'' \in M'(q', w)$ for some word $w \in A^*$, then $M(f_R(q'), w) \subseteq f_R(q'')$.*

Proof. For every $q \in f_R(q')$ there exists a word u such that $M(q_1, u) = q$ and $u \in P_r^{\mathscr{A}'}(q')$. Let $q_0 \in M(q, w)$. Then $q_0 \in M(q_1, uw)$ and $uw \in P_r^{\mathscr{A}'}(q'')$, which implies $q_0 \in f_R(q'')$ as required.

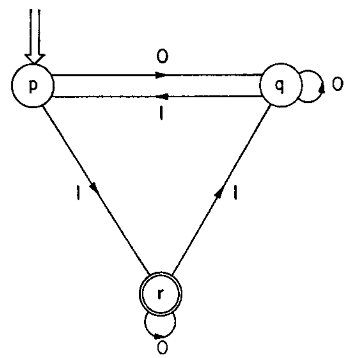

FIG. 6. The automaton $\tilde{\mathscr{A}}_0(R)$.

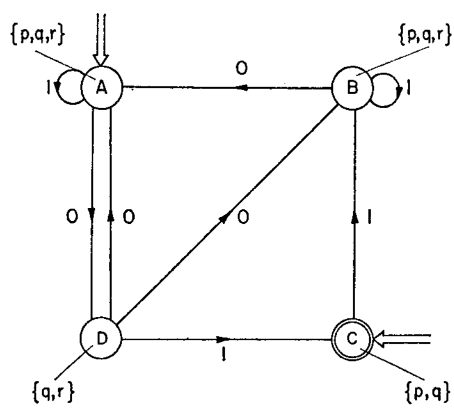

FIG. 7. The nondeterministic automaton $\mathscr{A}'$.

LEMMA 6.3. *Let $\mathscr{A}$, $\mathscr{A}'$ and R be as in Definition 6.4. If $\mathscr{A}'$ recognizes R, then for any final state $q' \in F'$ of $\mathscr{A}'$, $f_R(q') \subseteq F$.*

Proof. Let $q' \in F'$ and let $q \in f_R(q')$. Then there exists a word u such that $M(q_1, u) = q$ and $u \in P_r^{\mathscr{A}'}(q')$. The latter implies that u is accepted by $\mathscr{A}'$ and hence also by $\mathscr{A}$, i.e., $M(q_1, u) = q \in F$.

THEOREM 6.4. *Every nondeterministic automaton recognizing R is isomorphic to some partial automaton of $\mathscr{A}_k(R)$, for a large enough k.*

Proof. Let $\mathscr{A}' = (Q', M', Q_0', F')$ be a nondeterministic automaton recognizing R and let $\mathscr{A} = \tilde{\mathscr{A}}_0(R) = (Q, M, q_1, F)$. Define a mapping ρ from Q' into the set Q^k of states of the subset automaton $\mathscr{A}_k(R)$, where $k = \# Q'$, as follows:

$$\rho(q') = f_R(q')_{(i)}$$

where $1 \leqslant i \leqslant k$, and for any two states $q', q'' \in Q'$ such that $f_R(q') = f_R(q'')$, $\rho(q') \neq \rho(q'')$. Thus ρ is a 1-to-1 mapping; moreover, if $q_0' \in Q_0'$, then $q_1 \in f_R(q_0')$ and hence $\rho(q_0')$ is an initial state in $\mathscr{A}_k(R)$; similarly, using Lemma 6.3 one can see that ρ maps a final state of $\mathscr{A}'$ onto a final state of $\mathscr{A}_k(R)$. Furthermore, by Lemma 6.2, for every $a \in A$, $q'' \in M'(q', a)$ guarantees that $f_R(q'')_{(j)} \in M_k(f_R(q')_{(i)}, a)$ (for any $1 \leqslant i, j \leqslant k$). Thus every transition in $\mathscr{A}'$ corresponds to a transition in $\mathscr{A}_k(R)$. Hence ρ is an isomorphism from $\mathscr{A}'$ onto a partial automaton of $\mathscr{A}_k(R)$.

In many cases Theorem 6.4 can be utilized for determining the star height of a given event R. The technique will be illustrated by the following examples.

EXAMPLE 6.3. This example appears in McNaughton's paper [3]. The approach employed here is totally different from that used in [3], and throws a new light on the star height problem.

Let $\mathscr{A} = \tilde{\mathscr{A}}_0(R_1)$ be the reduced automaton considered in Example 6.1 (Fig. 4). Since $r(\mathscr{A}) = 2$, $h(R_1) \leqslant 2$ by Eggan's theorem. Now consider all partial automata of the subset automaton $\mathscr{A}_1(R_1)$ (Fig. 5) which have rank 1. By Proposition 6.1, all these recognize subsets of R_1. However, the partial automaton $\mathscr{A}'$ of $\mathscr{A}_1(R_1)$ (Fig. 8) recognizes the whole event R_1. Since $r(\mathscr{A}') = 1$ we deduce $h(R_1) = 1$.

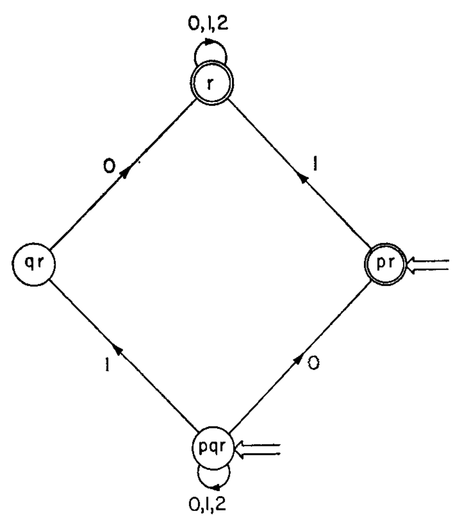

FIG. 8. The partial automaton $\mathscr{A}'$ of $\mathscr{A}_1(R_1)$.

EXAMPLE 6.4. Consider the reduced automaton $\mathscr{A} = \tilde{\mathscr{A}}_0(R_2)$ in Fig. 9. Clearly $r(\mathscr{A}) = 2$ and hence $h(R_2) \leqslant 2$. However, the partial automaton $\mathscr{A}'$ (Fig. 10) of $\mathscr{A}_1(R_2)$ is of rank 1 and equivalent to $\mathscr{A}$; thus a regular expression of apparent star height 1 (i.e., $E = (0 \cup 01 \cup 10 \cup 0111)^*1$) can be constructed for R_2 and $h(R_2) = 1$.

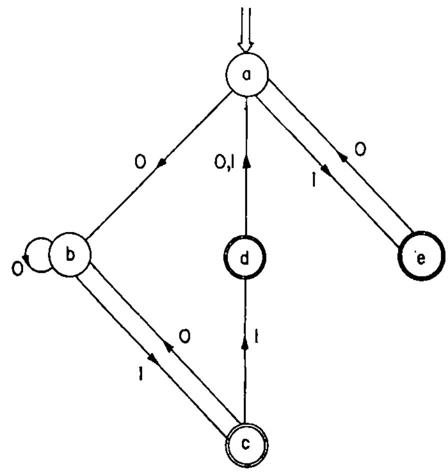

FIG. 9. The automaton $\tilde{\mathscr{A}}_0(R_2)$.

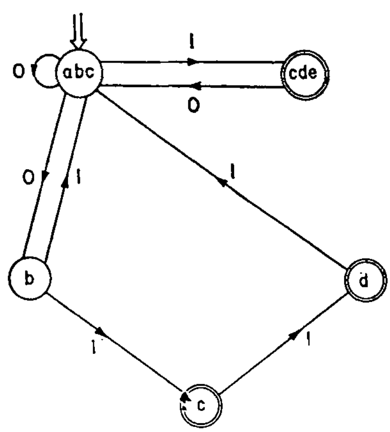

FIG. 10. The partial automaton $\mathscr{A}'$ of $\mathscr{A}_1(R_2)$.

Thus in many cases, there exists a partial automaton $\mathscr{A}'$ of the subset automaton of order 1, $\mathscr{A}_1(R)$, which recognizes R and whose rank equals the star height of R. Unfortunately, this is not always the case. The following is an example of an event R, for which there exists no partial automaton of $\mathscr{A}_1(R)$ recognizing R and of rank 1, but there exists a partial automaton of $\mathscr{A}_2(R)$ with the above properties.

EXAMPLE 6.5. Consider the automaton $\mathscr{A} = \tilde{\mathscr{A}}_0(R)$ in Fig. 11. It can be verified that no partial automaton of rank 1 of the subset automaton of order 1, $\mathscr{A}_1(R)$, recognizes the whole event R.

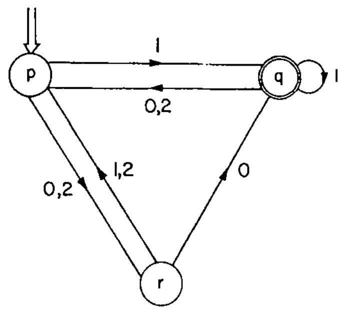

FIG. 11. The automaton $\mathscr{A} = \tilde{\mathscr{A}}_0(R)$.

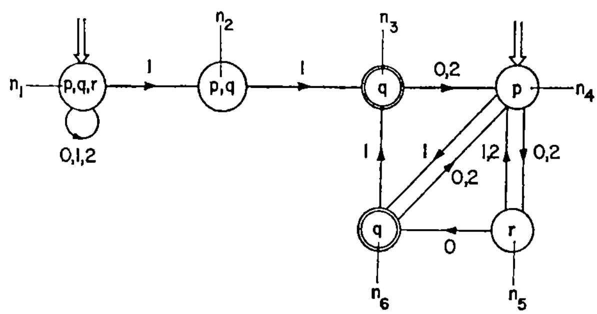

FIG. 12. The partial automaton $\mathscr{A}'$ of $\mathscr{A}_2(R)$.

Now consider the partial automaton $\mathscr{A}'$ of $\mathscr{A}_2(R)$ shown in Fig. 12. Clearly $r(\mathscr{A}') = 1$. To see that $\mathscr{A}'$ recognizes the whole event R, let $w \in R$, and consider the following two cases:

Case (a). w has no occurrence of two consecutive 1's in it. First note that the automaton $\mathscr{A}'' = \mathscr{A}' - [\{n_1, n_2\}]$ resembles $\mathscr{A}$, except for the missing self-loop on q labelled by 1, which is replaced by another "duplicate" of q and a 1-transition from q to its duplicate. This modification causes $\mathscr{A}''$ to have rank 1, even though $\mathscr{A}$ has rank 2. Now, since the subword 11 never occurs in w, the self-loop on q labelled by 1 in $\mathscr{A}$ will never be repeated two consecutive times; thus it can be replaced by the single 1-transition from q to its duplicate in $\mathscr{A}''$. Hence w will be recognized also by $\mathscr{A}'$.

Case (b). $w = w'\,11\,w''$, where w'' does not start with a 1 and has no occurrence of two consecutive 1's. Then let $\mathscr{A}'$ be started at state n_1, let w' take $\mathscr{A}'$ from state n_1 to itself, and then let the word 11 cause a transition in $\mathscr{A}'$ from state n_1 to n_3. Now since $w \in R$ and due to the fact that the word 11 maps all states of $\mathscr{A}$ to state q, w'' must map state q to itself in $\mathscr{A}$. Hence in $\mathscr{A}'$ state n_3 will be mapped by w'' to either n_3 or n_6. Thus w is accepted by $\mathscr{A}'$, and it follows that $h(R) = r(\mathscr{A}') = 1$.

As it turns out, for certain events R, even the examination of all partial automata of $\mathscr{A}_2(R)$ may not suffice for determining the star height of R. In fact, for any integer $t > 0$, an example of an event R_t can be constructed, such that no partial automaton of $\mathscr{A}_i(R_t)$, where $1 \leqslant i \leqslant t-1$, recognizes R_t and has rank $h(R_t)$; but there exists a partial automaton of $\mathscr{A}_t(R_t)$ recognizing R_t whose rank equals that star height of R_t. In other words, every minimum-rank nondeterministic automaton recognizing R_t has at least t states q_j, all of whose corresponding sets $f_{R_t}(q_j)$ are identical.

However, in those examples of R_t, the number of states of $\mathscr{A}_0(R_t)$ grows with t. It would seem that for a fixed number of states, say m_0, of the reduced automaton $\mathscr{A}_0(R)$, there exists an integer $k_0 = k_0(m_0) > 0$ such that the subset automaton $\mathscr{A}_{k_0}(R)$ has a partial automaton $\mathscr{A}'$ recognizing R and of rank $h(R)$. This would lead to the existence of an algorithm for determining the star height of regular events.

Remark. The interested reader is referred to [12] in which further properties of star height are presented.

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